

Chapter 10

Electrodynamics

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We compute the electric fields due to static and moving charges, describe methods for computing the electric potential in boundary value problems, and solve Maxwell's equations numerically.

10.1 Static Charges

Suppose we want to know the electric field $\mathbf{E}(\mathbf{r})$ at the point $\mathbf{r}$ due to N point charges $q_1, q_2, \dots, q_N$ at fixed positions $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$. We know that $\mathbf{E}(\mathbf{r})$ satisfies a superposition principle and is given by

$$\mathbf{E}(\mathbf{r}) = K \sum_i^N \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i), \quad (10.1)$$

where $\mathbf{r}_i$ is the fixed location of the i th charge and K is a constant that depends on our choice of units. One of the difficulties associated with electrodynamics is the competing systems of units. In the SI (or rationalized MKS) system of units, the charge is measured in coulombs (C) and the constant K is given by

$$K = \frac{1}{4\pi\epsilon_0} \approx 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2. \quad (\text{SI units}) \quad (10.2)$$

The constant ϵ_0 is the electrical permittivity of free space. This choice of units is not convenient for computer programs because $K \gg 1$. Another popular system of units is the Gaussian (cgs) system for which the constant K is absorbed into the unit of charge so that $K = 1$. Charge is in electrostatic units or esu. One feature of Gaussian units is that the electric and magnetic fields

have the same units. For example, the (Lorentz) force on a particle of charge q and velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ has the form

$$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}). \quad (\text{Gaussian units}) \quad (10.3)$$

These virtues of the Gaussian system of units lead us to adopt this system for this chapter even though SI units are used in introductory texts.

10.2 Electric Fields

The electric field is an example of a *vector field* because it defines a vector quantity at every point in space. One way to visualize this field is to divide space into a discrete grid and to draw arrows in the direction of $\mathbf{E}$ at the vertices of this grid. The length of the arrow can be chosen to be proportional to the magnitude of the electric field. Another possibility is to use color or gray scale to represent the magnitude. The Vector Field element supports both representations (see Appendix 10A) but we usually use an arrow's color rather than its length to represent field strength because it produces a more effective representation of vector fields over a wider dynamic range.

The Point Charge Electric Field model in this chapter's source code directory computes the electric field due to multiple point charges. The particle position (x_i, y_i) and charge q_i are stored in 10-element arrays but fewer than this maximum number are usually displayed. Only the first n particles are visible and only these active particles are used to compute the field. The Add Particle button in the control panel increments the number of particles and the Reset button removes all charges. Whenever a charge is added or moved, the electric field is recomputed using a fixed relation.

Because users can change the number of active particles and their location by dragging or by entering values into text fields, we have defined custom methods to respond to these actions. Dragging the i -th particle (circle) automatically changes the array element that is bound (connected) to that circle. Because number fields for x_i , y_i and q_i can only establish bi-directional communication with primitive data types, the drag action for the particles (circles) invokes a custom method that sets variable bound to the number field from array elements.

```
// action method invoked when dragging a particle
public void setNumbersFromArray () {
    xs=x[index]; // xs is bound to control panel number field
    ys=y[index]; // ys is bound to control panel number field
    qs=q[index]; // qs is bound to control panel number field
}
```

The `setArrayFromNumbers` custom method reverses the process and sets an array element when a number field is edited. You will modify Point Charge Electric Field model to include a moving test charge in Problem 10.1.

Listing 10.1: The fixed relation computes the electric field at each Vector Field grid point.

```
// the electric field is computed at each grid point
for(int i=0; i<eGrid; i++){ // loop over grid x
    double xi= _view.vectorField.indexToX(i); // grid point x-coordinate
```

```

for(int j=0; j<eGrid; j++){           // loop over grid y
    double yj= _view.vectorField.indexToY(j); // grid point y-coordinate
    Ex[i][j]=Ey[i][j]=0;              // zero field at point
    for(int p=0; p<n; p++){           // loop over charges
        double dx=xi-x[p];            // grid point x-separation
        double dy=yj-y[p];            // grid point y-separation
        double r2=dx*dx+dy*dy;         // distance squared
        double r3=r2*Math.sqrt(r2);   // distance
        if(r2!=0){                    // check for singularity
            Ex[i][j] += q[p]*dx/r3;    // add E field x-component
            Ey[i][j] += q[p]*dy/r3;    // add E field y-component
        }                             // end of if statement
    }                                 // end of charge loop
}                                   // end of y loop
}                                   // end of x loop

```

Problem 10.1. Motion of a charged particle in an electric field

- a. Test the Point Charge Electric Field model by adding one charge at a time at various locations. Do the electric field patterns look reasonable? For example, does the electric field point away from positive charges and toward negative charges? How well is the magnitude of the electric field represented?
- b. Modify the Point Charge Electric Field model so that it uses an evolution page to compute the motion of a test particle of mass m and charge q in the presence of the electric field created by a fixed distribution of point charges. That is, create a drawable test charge that implements the ODE interface and add it to the vector field frame. Use the same approach that was used for the trajectory problems in Chapter 5. The acceleration of the charge is given by $q\mathbf{E}/m$, where $\mathbf{E}$ is the electric field due to the fixed point charges. Use a higher-order algorithm to advance the position and velocity of the particle. (Ignore the effects of radiation due to accelerating charges.)
- c. Assume that $\mathbf{E}$ is due to a charge $q(1) = 1.5$ fixed at the origin. Simulate the motion of a charged particle of mass $m = 0.1$ and charge $q = 1$ initially at $x = 1, y = 0$. Consider the following initial conditions for its velocity: $v_x = 0, v_y = 0$; $v_x = 1, v_y = 0$; $v_x = 0, v_y = 1$; and $v_x = -1, v_y = 0$. Is the trajectory of the particle tangent to the field vectors? Explain.
- d. Assume that the electric field is due to two fixed point charges: $q(1) = 1$ at $\mathbf{x}(1) = 2, y(1) = 0$ and $q(2) = -1$ at $\mathbf{x}(2) = -2, y(2) = 0$. Place a charged particle of unit mass and unit positive charge at $x = 0.05, y = 0$. What do you expect the motion of this charge to be? Do the simulation and determine the qualitative nature of the motion.
- e.* Consider the motion of a charged particle in the vicinity of the electric dipole defined in part (d). Choose the initial position to be five times the separation of the charges in the dipole. Do you find any bound orbits? Do you find any closed orbits or do all orbits show some precession?

■

10.3 Electric Field Lines

Another way of visualizing the electric field is to draw *electric field lines*. The properties of these lines are as follows:

1. An electric field line is a directed line whose tangent at every position is parallel to the electric field at that position.
2. The lines are smooth and continuous except at singularities such as point charges. (It makes no sense to talk about the electric field *at* a point charge.)
3. The density of lines at any point in space is proportional to the magnitude of the field at that point. This property implies that the total number of electric field lines from a point charge is proportional to the magnitude of that charge. The value of the proportionality constant is chosen to provide the clearest pictorial representation of the field. The drawing of field lines is art plus science.

The Electric Field Line model draws electric field lines in two dimensions. The model implements the following algorithm:

1. Begin at a point (x, y) and compute the components E_x and E_y of the electric field vector $\mathbf{E}$ using (10.1).
2. Draw a small line segment of size $\Delta s = |\Delta \mathbf{s}|$ tangent to $\mathbf{E}$ at that point. The components of the line segment are given by

$$\Delta x = \Delta s \frac{E_x}{|\mathbf{E}|} \text{ and } \Delta y = \Delta s \frac{E_y}{|\mathbf{E}|}. \quad (10.4)$$

3. Iterate the process beginning at the new point $(x + \Delta x, y + \Delta y)$. Continue until the field line approaches a point charge singularity or escapes toward infinity.

This field line algorithm is equivalent to solving the following differential equations:

$$\frac{dx}{ds} = \frac{E_x}{|\mathbf{E}|} \quad (10.5a)$$

$$\frac{dy}{ds} = \frac{E_y}{|\mathbf{E}|}. \quad (10.5b)$$

Because a field line extends in both directions from the algorithm's starting point, the computation must be repeated in the $(-E_x/|\mathbf{E}|, -E_y/|\mathbf{E}|)$ direction to obtain a complete visualization of the field line.

The Electric Field Lines model implements this algorithm but note that this model draws a field line in response to a mouse click and does not draw a collection of field lines with a density proportional to the field intensity. The plotting panel action method the `computeFieldLine` custom method and this method explicitly advances the ODE solver 10,000 steps or until the field line approaches a point charge. Listing 10.2 shows how this computation is done.

Listing 10.2: The `computeFieldLine` method computes an electric field line by explicitly advancing the ODE solver.

```
public void computeFieldLine () {
    atSingularity=false;
    // reset ODE event
    org.opensourcephysics.numerics.EJSODE ode =
        _getODE("Field Line Evolution");           // ODE solver
    xFL=_view.electricFieldPlottingPanel.getMouseX(); // mouse x-location
    yFL=_view.electricFieldPlottingPanel.getMouseY(); // mouse y-location
    _view.fieldLineTrail.moveToPoint(xFL,yFL);      // starting point
    for(int i=0;i<10000; i++) {
        ode.step();                                // advance the ODE
        _view.fieldLineTrail.addPoint(xFL,yFL);    // add data point
        if(atSingularity) break;                   // break from loop
    }
    atSingularity=false;                           // reset ODE event
    direction = -direction;                         // change direction
    xFL=_view.electricFieldPlottingPanel.getMouseX(); // mouse x-location
    yFL=_view.electricFieldPlottingPanel.getMouseY(); // mouse y-location
    _view.fieldLineTrail.moveToPoint(xFL,yFL);      // starting point
    for(int i=0;i<10000; i++) {
        ode.step();                                // advance the ODE
        _view.fieldLineTrail.addPoint(xFL,yFL);    // add data
        if(atSingularity) break;                   // break from loop
    }
}
```

The method in Listing 10.2 starts by resetting a boolean variable that changes its value if the field line approaches a singularity. The `ode` variable in the second line gives us direct access to the model's internal ODE solver as described in Appendix 5A. The custom method then sets the dynamical variables to the mouse-click position and moves the trail to the start of the field line computation. We now advance the field line ODE using a loop but exit the loop if the computation approaches a singularity. (Examine the Singularity event on the evolution workpanel to see how this is done.) This field line computation is then repeated to draw the field line from the mouse click in the negative direction. Direct access to the ODE solver produces a very fast computation because it does not evaluate fixed relations or automatically collect data. Most field lines are computed faster than the typical user's mouse-click reaction time. Note that we explicitly add data to the field line trail after every ODE step and that this data is displayed when the action method completes and the model's view is updated.

Problem 10.2. Verification of field line model

- Draw field lines for a few simple sets of one, two, and three charges. Choose sets of charges for which all have the same sign, and sets for which they are different. Verify that the field lines never connect charges of the same sign. Why do field lines never cross? Are the units of charge and distance relevant?
- Compare the Electric Field Line model and Point Charge Electric Field model. Which representation conveys more information? Consider how each model provides (or does not provide)

information about the electric field magnitude and direction. Discuss some of the difficulties with making an accurate field line diagram.

- c. The Electric Field Line model uses a constant value for Δs . Modify the algorithm so that the calculation continues when a field line moves off the screen, but speed up the algorithm by increasing the value of Δs . ■

Problem 10.3. Electric field lines from point charges

- a. Modify the Electric Field Line model so that ten field lines per unit of charge are computed whenever a new charge is added to the panel or when a charge is moved. Start these field lines close to each positive charge in such a way that they propagate away from the charge. Should you start these field lines on both positive and negative charges? Explain your answer.
- b. Draw the field lines for an electric dipole.
- c. Draw the field lines for the electric quadrupole with $q(1) = 1$, $x(1) = 1$, $y(1) = 1$, $q(2) = -1$, $x(2) = -1$, $y(2) = 1$, $q(3) = 1$, $x(3) = -1$, $y(3) = -1$, and $q(4) = -1$, $x(4) = 1$, and $y(4) = -1$.
- d. A continuous charge distribution can be approximated by a large number of closely spaced point charges. Draw the electric field lines due to a row of ten equally spaced unit charges located between -2.5 and $+2.5$ on the x axis. How does the electric field distribution compare to the distribution due to a single point charge?
- e. Repeat part (c) with two rows of equally spaced positive charges on the lines $y = 0$ and $y = 1$, respectively. Then consider one row of positive charges and one row of negative charges. ■

Problem 10.4. Field lines due to infinite line of charge

- a. The the Electric Field Line model plots field lines in two dimensions. Sometimes this restriction can lead to spurious results (see Freeman). Consider four identical charges placed at the corners of a square. Use the model to plot the field lines. What, if anything, is wrong with the results? What should happen to the field lines near the center of the square?
- b. The two-dimensional analog of a point charge is an infinite line (thin cylinder) of charge perpendicular to the plane. The electric field due to an infinite line of charge is proportional to the linear charge density and inversely proportional to the distance (instead of the distance squared) from the line of charge to a point in the plane. Modify the Electric Field Line model to compute the field lines from line charges with $E(r) = 1/r$. Use your modified class to draw the field lines due to four identical line charges located at the corners of a square, and compare the field lines with your results in part (a).
- c. Use your modified model from part (b) to draw the field lines for the two-dimensional analogs of the distributions considered in Problem 10.3. Compare the results for two and three dimensions, and discuss any qualitative differences.
- d. Can your model be used to demonstrate Gauss's law using point charges? What about line charges? ■

10.4 Electric Potential

It often is easier to analyze the behavior of a system using energy rather than force concepts. We define the electric potential $V(\mathbf{r})$ by the relation

$$V(\mathbf{r}_2) - V(\mathbf{r}_1) = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{r}, \quad (10.6)$$

or

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}). \quad (10.7)$$

Only differences in the potential between two points have physical significance. The gradient operator ∇ is given in Cartesian coordinates by

$$\nabla = \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}}, \quad (10.8)$$

where the vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along the x , y , and z axes respectively. If V depends only on the magnitude of $\mathbf{r}$, then (10.7) becomes $E(r) = -dV(r)/dr$. Recall that $V(r)$ for a point charge q relative to a zero potential at infinity is given by

$$V(r) = \frac{q}{r}. \quad (\text{Gaussian units}) \quad (10.9)$$

The surface on which the electric potential has an equal value everywhere is called an *equipotential surface* (a curve in two dimensions). The Scalar Field element provides a graphical representations of electrostatic potential fields (see Appendix 9B) because the scalar field is constant along a contour line. Constant color can also be used to indicate regions of uniform potential and we usually color regions of high potential red and regions of low potential blue. Problem 10.5 uses the Scalar Field element to study electric potential from point charges.

Problem 10.5. Equipotential contours

- Create a Point Charge Potential model based on Point Charge Electric Field model that draws equipotential lines using a Scalar Field element. Test your model using the charge distributions considered in Problem 10.3.
- Explain why equipotential surfaces (lines in two dimensions) never cross. ■

Because $\mathbf{E}$ is in the direction in which the electric potential decreases most rapidly, the electric field lines are orthogonal to the equipotential surfaces at any point. We can use this orthogonality between the electric field lines and the equipotential lines to modify Electric Field Line model so that it draws the latter. Because the components of the line segment $\Delta \mathbf{s}$ parallel to the electric field line are given by $\Delta x = \Delta s(E_x/E)$ and $\Delta y = \Delta s(E_y/E)$, the components of the line segment perpendicular to $\mathbf{E}$, and hence parallel to the equipotential line, are given by $\Delta x = -\Delta s(E_y/E)$ and $\Delta y = \Delta s(E_x/E)$. It is unimportant whether the minus sign is assigned to the x or y component, because the only difference would be the direction that the equipotential lines are drawn.

Problem 10.6. Equipotential lines

- a. Create a model that is based on the Electric Field Line model to draw equipotential lines for the charge distributions considered in Problem 10.3. Use a mouse click to determine the starting point of the equipotential line computation. The computation should stop when the line returns close to the starting point or after an unreasonable number of calculations. Will a potential line always close on itself?
- b. What would a higher density of equipotential lines mean if we drew lines such that each adjacent line differed from a neighboring one by a fixed potential difference?
- c. Explain why equipotential surfaces never cross. ■

Problem 10.7. The electric potential due to a finite sheet of charge

Consider a uniformly charged nonconducting plate of total charge Q and linear dimension L centered at $(0,0,0)$ in the x - y plane. In the limit $L \rightarrow \infty$ with the charge density $\sigma = Q/L^2$ a constant, we know that the electric field is normal to the sheet and its magnitude is given by $2\pi\sigma$ (Gaussian units). What is the electric field due to a finite sheet of charge? A simple method is to divide the plate into a grid of p square regions on a side such that each region is sufficiently small to be approximated by a point charge of magnitude $q = Q/p^2$. Because the potential is a scalar, it is easier to compute the total potential rather than the total electric field from the $N = p^2$ point charges. Use the relation (10.9) for the potential from a point charge and create a model to compute $V(z)$ and hence $E_z = -\partial V(z)/\partial z$ for points along the z -axis and perpendicular to the sheet. Take $L = 1$, $Q = 1$, and $p = 10$ for your initial calculations. Increase p until your results for $V(z)$ do not change significantly. Plot $V(z)$ and E_z as a function of z and compare their z -dependence to their infinite sheet counterparts. ■

***Problem 10.8.** Electrostatic shielding

We know that the (static) electric field is zero inside a conductor, all excess charges reside on the surface of the conductor, and the surface charge density is greatest at the points of greatest curvature. Although these properties are plausible, it is instructive to do a simulation to see how these properties follow from Coulomb's law. For simplicity, consider the conductor to be two-dimensional so that the potential energy is proportional to $\ln r$ rather than $1/r$ (see Problem 10.4). It also is convenient to choose the surface of the conductor to be an ellipse.

- a. If we are interested only in the final distribution of the charges and not in the dynamics of the system, we can use a Monte Carlo method. Our goal is to find the minimum energy configuration beginning with the N charges randomly placed within a conducting ellipse. One method is to choose a charge i at random, and make a trial change in the position of the charge. The trial position should be no more than δ from the old position and still be within the ellipse. Choose $\delta \approx b/10$, where b is the semiminor axis of the ellipse. Compute the change in the total potential energy given by (in arbitrary units)

$$\Delta U = - \sum_j [\ln r_{ij}^{(\text{new})} - \ln r_{ij}^{(\text{old})}]. \quad (10.10)$$

The sum is over all charges in the system not including i . If $\Delta U > 0$, then reject the trial move, otherwise accept it. Repeat this procedure many times until very few trial moves are accepted. Create a model to implement this Monte Carlo algorithm. Run the simulation for

$N \geq 20$ charges inside a circle and then repeat the simulation for an ellipse. How are the charges distributed in the (approximately) minimum energy distribution? Which parts of the ellipse have a higher charge density?

- b. Repeat part (a) for a two-dimensional conductor, but assume that the potential energy $U \sim 1/r$. Do the charges move to the surface?
- c. Is it sufficient that the interaction be repulsive for the results of parts (a) and (b) to hold?
- d. Repeat part (a) with the added condition that there is a fixed positive charge of magnitude $N/2$ located outside the ellipse. How does this fixed charge effect the charge distribution? Are the excess free charges still at the surface? Try different positions for the fixed charge.
- e. Repeat parts (a) and (b) for $N = 50$ charges located within an ellipsoid in three dimensions. ■

10.5 Numerical Solutions of Boundary Value Problems

In Section 10.1 we found the electric fields and potentials due to a fixed distribution of charges. Suppose that we do not know the positions of the charges and instead know only the potential on a set of boundaries surrounding a charge-free region. This information is sufficient to determine the potential $V(\mathbf{r})$ at any point within the charge-free region.

The direct method of solving for $V(x, y, z)$ is based on Laplace's equation which can be expressed in Cartesian coordinates as

$$\nabla^2 V(x, y, z) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (10.11)$$

The problem is to find the function $V(x, y, z)$ that satisfies (10.11) and the specified boundary conditions. This type of problem is an example of a *boundary value* problem. Because analytical methods for regions of arbitrary shape do not exist, the only general approach is to use numerical methods. Laplace's equation is not a new law of physics, but can be derived directly from (10.7) and the relation $\nabla \cdot \mathbf{E} = 0$ or indirectly from Coulomb's law in regions of space where there is no charge.

For simplicity, we consider only two-dimensional boundary value problems for $V(x, y)$. We use a finite difference method and divide space into a discrete grid of sites located at the coordinates (x, y) . In Problem 10.9b, we show that in the absence of a charge at (x, y) , the discrete form of Laplace's equation satisfies the relation

$$V(x, y) \approx \frac{1}{4}[V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)], \quad (\text{two dimensions}) \quad (10.12)$$

where $V(x, y)$ is the value of the potential at the site (x, y) . Equation (10.12) says that $V(x, y)$ is the average of the potential of its four nearest neighbor sites. This remarkable property of $V(x, y)$ can be derived by approximating the partial derivatives in (10.11) by finite differences (see Problem 10.9b).

Problem 10.9. Verification of the difference equation for the potential Verify (10.12) by calculating the potential due to a point charge at a point in space selected by the user and at the four nearest neighbors. As the form of (10.12) implies, the average of the potential at the four neighboring sites should equal the potential at the center site. We assume the form (10.9) for the potential $V(r)$ due to a point charge, a form that satisfies Laplace's equation for $r \neq 0$.

- Modify the Point Charge Potential model that you created in Section 10.4 to compare the computed potential at a point to the average of the potential at its four nearest neighbor sites. Choose reasonable values for the spacings Δx and Δy and consider a point that is not too close to a source charge. Do similar measurements for other points. Does the relative agreement with (10.12) depend on the distance of the point to the source charge? Choose smaller values of Δx and Δy and determine if your results are in better agreement with (10.12). Does it matter whether Δx and Δy have the same value?
- Derive the finite difference equation (10.12) for $V(x, y)$ using the second-order Taylor expansion:

$$V(x + \Delta x, y) = V(x, y) + \Delta x \frac{\partial V(x, y)}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V(x, y)}{\partial x^2} + \dots \quad (10.13)$$

$$V(x, y + \Delta y) = V(x, y) + \Delta y \frac{\partial V(x, y)}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 V(x, y)}{\partial y^2} + \dots \quad (10.14)$$

The effect of including higher derivatives is discussed by MacDonald (see references). ■

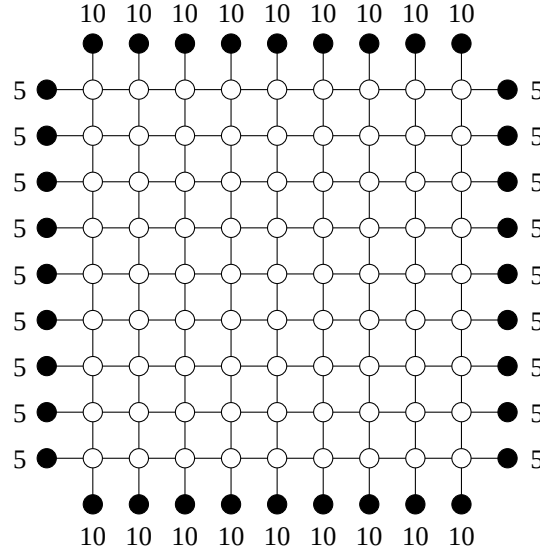


Figure 10.1: Potential distribution considered in Problem 10.10c. The number of interior sites in each direction is nine.

Now that we have found that (10.12), a finite difference form of Laplace's equation, is consistent with Coulomb's law, we adopt (10.12) as the basis for computing the potential for systems for

which we cannot calculate the potential directly. In particular, we consider problems where the potential is specified on a closed surface that divides space into interior and exterior regions in which the potential is independently determined. For simplicity, we consider only two-dimensional geometries. The approach, known as the *relaxation method*, is based on the following algorithm:

1. Divide the region of interest into a rectangular grid spanning the region. The region is enclosed by a surface (curve in two dimensions) with specified values of the potential along the curve.
2. Assign to a boundary site the potential of the boundary nearest the site.
3. Assign all interior sites an arbitrary potential (preferably a reasonable guess).
4. Compute new values for the potential V for each interior site. Each new value is obtained by finding the average of the previous values of the potential at the four nearest neighbor sites.
5. Repeat step (4) using the values of V obtained in the previous iteration. This iterative process is continued until the potential at each interior site is computed to the desired accuracy.

The Laplace Equation model in this chapter's source code directory implements the Jacobi method using a grid of voltages and a boolean grid to signal the presence of a conductor. Open this model and examine the evolution page. The disabled Gauss-Seidel evolution page will be used in Problem 10.11.

Listing 10.3: The Jacobi relaxation evolution page implements an iterative relaxation algorithm to solve Laplace's Equation.

```
double V[][] = new double[ gridSize ][ gridSize ];
for(int i = 1; i < gridSize - 1; i++) {
    for(int j = 1; j < gridSize - 1; j++) {
        if(isConductor[i][j]){
            V[i][j] = potential[i][j];
        } else { // compute voltage for non-conductors
            V[i][j] = (potential[i-1][j] + potential[i+1][j] +
                      potential[i][j-1] + potential[i][j+1]) / 4;
        }
    }
}
double error = 0;
for(int i = 1; i < gridSize - 1; i++) {
    for(int j = 1; j < gridSize - 1; j++) {
        error = Math.max(error, Math.abs(potential[i][j] - V[i][j]));
        potential[i][j] = V[i][j];
    }
}
if(error < maximumError) {
    _pause();
    msg = "Computation done.";
}
```

As the algorithm loops through the grid sites, it first checks if each grid site is a conductor. If it is, the potential is that of the conductor. If not, a new potential is calculated and assigned to the **potential** array element. A local variable named **error** keeps track of the maximum change in potential and this maximum change is compared to the global **maximumError** variable to determine the end of the evolution. In Problems 10.10–10.12 you are asked to modify Laplace Relaxation model to compute the potential for various geometries.

Problem 10.10. Numerical solution of the potential within a rectangular region

- a. Modify Laplace Equation model to determine the potential $V(x, y)$ in a square region with linear dimension $L = 10$. The boundary of the square is at a potential $V = 10$. Choose the grid size $\Delta x = \Delta y = 1$. Before you run the simulation, guess the exact form of $V(x, y)$ and set the initial values of the interior potential 10% lower than the exact answer. How many iterations are necessary to achieve 1% accuracy? Decrease the grid size by a factor of two, and determine the number of iterations that are now necessary to achieve 1% accuracy.
- b. Consider the same geometry as in part (a), but set the initial potential at the interior sites equal to zero except for the center site whose potential is set equal to four. Does the potential distribution evolve to the same values as in part (a)? What is the effect of a poor initial guess? Are the final results independent of your initial guess?
- c. Modify the Laplace Equation model so that the value of the potential at the four sides is 5, 10, 5, and 10, respectively (see Figure 10.1). Sketch the equipotential surfaces. What happens if the potential is 10 on three sides and 0 on the fourth? Start with a reasonable guess for the initial values of the potential at the interior sites and iterate until 1% accuracy is obtained.
- d.* Consider the same initial choice of the potential as in part (b) and focus your attention on the potential at the sites near the center of the square. If the central site has an initial potential of four, what is the potential at the nearest neighbor sites after the first iteration? Follow the distribution of the potential as a function of the number of iterations and verify that the nature of the relaxation of the potential to its correct distribution is closely related to *diffusion* (see Chapter 7). It may be helpful to increase the number of sites in the grid and the initial value of the potential at the central site to see the nature of the relaxation more clearly. ■

In Problem 10.10, we studied a simple version of the relaxation method known as the Jacobi method. In particular, the new potential at each site is based on the values of the potentials at the neighboring sites at the previous iteration. After the entire lattice is visited, the potential at each site is updated *simultaneously*. The difficulty with this relaxation method is that it converges very slowly. The use of more general relaxation methods is discussed in many texts (cf. Sadiku or Press et al.). In Problem 10.11 we consider a method known as *Gauss-Seidel* relaxation.

Problem 10.11. Gauss-Seidel relaxation

- a. Disable the Jacobi relaxation method and enable the Gauss Seidel method in the model that you used in Problem 10.10 so that the potential at each site is updated sequentially. That is, after the average potential of the nearest neighbor sites of site i is computed the potential at i is updated immediately. In this way the new potential of the next site is computed using the most recently computed values of its nearest neighbor potentials. Are your results better, worse, or about the same as for the simple relaxation method?

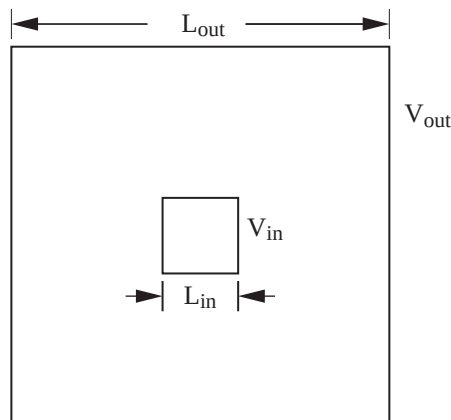


Figure 10.2: The geometry of the two concentric squares considered in Problem 10.12.

- b. Imagine coloring the alternate sites of a grid red and black, so that the grid resembles a checkerboard. Modify the model so that all the red sites are updated first, and then all the black sites are updated. This ordering is repeated for each iteration. Do your results converge any more quickly than in part (a)?
- c.* The slow convergence of the relaxation methods we have explored is due to the fact that it takes a long time for a change in the potential at one site to effect changes further away. We can improve the Gauss-Seidel method by using an *overrelaxation* method that updates the new potential as follows:

$$V_{\text{new}}(x, y) = wV_{\text{ave}}(x, y) + (1 - w)V(x, y), \quad (10.15)$$

where $V_{\text{ave}}(x, y)$ is the average of the potential of the four neighbors of (x, y) . The overrelaxation parameter w is in the range $1 < w < 2$. The effect of w is to cause the potential to change by a greater amount than in the simple relaxation procedure. Explore the dependence of the rate of convergence on w . A relaxation method that increases the rate of convergence is explored in Project 10.17. ■

Problem 10.12. The capacitance of concentric squares

- a. Use a relaxation method to compute the potential distribution between the two concentric square cylinders shown in Figure 10.2. The potential of the outer square conductor is $V_{\text{out}} = 10$ and the potential of the inner square conductor is $V_{\text{in}} = 5$. The linear dimensions of the exterior and interior squares are $L_{\text{out}} = 25$ and $L_{\text{in}} = 5$, respectively. Modify your model so that the potential of the interior square is fixed. Sketch the equipotential surfaces.
- b. A system of two conductors with charge Q and $-Q$ respectively has a capacitance C that is defined as the ratio of Q to the potential difference ΔV between the two conductors. Determine the capacitance per unit length of the concentric cylinders considered in part (a). In this case $\Delta V = 5$. The charge Q can be determined from the fact that near a conducting surface, the surface charge density σ is given by $\sigma = E_n/4\pi$, where E_n is the magnitude of the electric

field normal to the surface. E_n can be approximated by the relation $-\delta V/\delta r$, where δV is the potential difference between a boundary site and an adjacent interior site a distance δr away. Use the result of part (a) to compute δV for each site adjacent to the two square surfaces. Use this information to determine E_n for the two surfaces and the charge per unit length on each conductor. Are the charges equal and opposite in sign? Compare your numerical result to the capacitance per unit length, $1/2 \ln r_{\text{out}}/r_{\text{in}}$, of a system of two concentric circular cylinders of radii r_{out} and r_{in} . Assume that the circumference of each cylinder equals the perimeter of the corresponding square, that is, $2\pi r_{\text{out}} = 4L_{\text{out}}$ and $2\pi r_{\text{in}} = 4L_{\text{in}}$.

- c. Move the inner square 1 cm off center and repeat the calculations of parts (a) and (b). How do the potential surfaces change? Is there any qualitative difference if we set the inner conductor potential equal to -5 ? ■

Laplace's equation holds only in charge-free regions. If there is a charge density $\rho(x, y, z)$ in the region, we need to use *Poisson's equation* which can be written as

$$\nabla^2 V(\mathbf{r}) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho(\mathbf{r}), \quad (10.16)$$

where $\rho(\mathbf{r})$ is the charge density. The difference form of Poisson's equation is given in two dimensions by

$$\begin{aligned} V(x, y) \approx & \frac{1}{4} [V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)] \\ & + \frac{1}{4} \Delta x \Delta y 4\pi\rho(x, y). \end{aligned} \quad (10.17)$$

Note that the product $\rho(x, y)\Delta x\Delta y$ is the total charge in a $\Delta x \times \Delta y$ region centered at (x, y) .

Problem 10.13. Surface charge

- Poisson's equation can be used to find the surface charge on a conductor after Laplace's equation has been solved. The potential is fixed at the boundary sites. If we assume the boundary is a conductor with some thickness then we can assume that the potential for the next layer of sites outside the boundary has the same potential as the boundary. If we use this assumption, then after we have solved numerically for the potential of the interior sites, we will find that the average value of the neighbors of a boundary site will not equal the imposed potential. From (10.17) the difference will equal $\Delta x \Delta y \pi \rho(x, y)$. Modify the Laplace Equation model to calculate and display the surface charge density assuming $\Delta x = \Delta y = 1$. Notice that because we are in two dimensions the "surface" charge density, $\Delta x \Delta y \rho(x, y)$, is a linear density of charge per unit length.
- Using the same system as in Problem 10.10.c find the surface charge density on all boundary sites. Make a reasonable choice for assigning the potential at the corner sites.
- Model a system with the boundary at a potential $V = 0$ and a centered interior rectangle of 6×12 at a potential of $V = 10$. Where is the charge density the highest?
- Repeat if the interior rectangle is placed close to an edge. ■

Problem 10.14. Numerical solution of Poisson's equation

- Consider a square of linear dimension $L = 25$ whose boundary is fixed at a potential equal to $V = 10$. Assume that the interior region has a uniform charge density ρ such that the total charge is $Q = 1$. Modify Laplace Equation model to compute the potential distribution for this case. Compare the equipotential surfaces obtained for this case to that found in Problem 10.12.
- Find the potential distribution if the charge distribution of part (a) is restricted to a 5×5 square at the center.
- Find the potential distribution if the charge distribution of part (a) is restricted to a 1×1 square at the center. How does the potential compare to that of a point charge without the boundary?

■

***Problem 10.15.** Vector potential and magnetic fields

The magnetic field from arbitrary currents also can be obtained using Poisson's equation. The field is generated from a vector potential $\mathbf{A}$ that satisfies

$$\nabla^2 \mathbf{A} = \mu \mathbf{j} \quad (10.18)$$

where $\mathbf{j}$ is the current density in the wires and μ is the magnetic permeability. If current flows only in the z direction, then $\mathbf{j} = (0, 0, j_z(x, y))$ and $\mathbf{A} = (0, 0, A_z(x, y))$, and we again have a two-dimensional problem that can be solved using the relaxation method.

Create a models that displays the magnetic field from an arbitrary number of wires. Combine features of the Point Charge Electric Field and the Laplace Equation models. The computation is performed using the Evolution workpanel to perform a Gauss-Seidel relaxation step. Compute the magnetic field after the computation converges by computing the curl of the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (10.19)$$

See (??) for how to compute the curl when only discrete values are available.

Dielectrics can be added to the solution of Laplace's equation by adding an array to store the dielectric constant k at every grid site and imposing the condition:

$$D_{1n} = D_{2n}, \quad (10.20)$$

where $\mathbf{D} = k\mathbf{E}$ and k is the dielectric susceptibility. This condition is equivalent to

$$0 = \oint_l k \nabla V \cdot d\mathbf{l} = \oint_l k \frac{\partial V}{\partial n} dl, \quad (10.21)$$

where $\partial V / \partial n$ denotes the derivative of V normal to the contour l . If we approximate (10.21) along each edge of length $2h$ using a finite difference for the derivative, we obtain

$$0 = k_1 \frac{V_1 - V_0}{h} 2h + k_2 \frac{V_2 - V_0}{h} 2h + k_3 \frac{V_3 - V_0}{h} 2h + k_4 \frac{V_4 - V_0}{h} 2h. \quad (10.22)$$

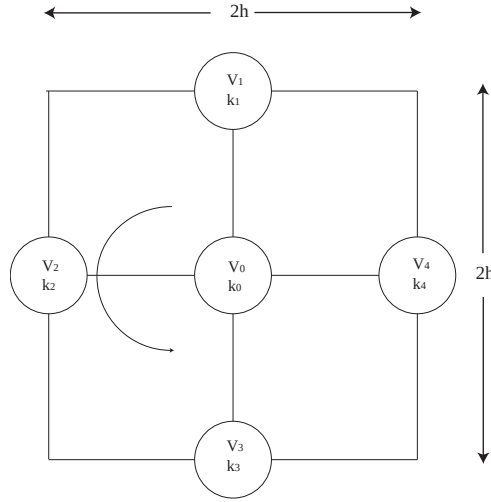


Figure 10.3: The path integral around the grid sites $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4$ is based on Gauss's law for the electric field.

We rearrange terms in (10.22) and find a modified form of (10.12) that includes the dielectric

$$V_0 = \frac{1}{4(k_1 + k_2 + k_3 + k_4)} [k_1 V_1 + k_2 V_2 + k_3 V_3 + k_4 V_4], \quad (10.23)$$

where k_i is the average dielectric constant at a site where the electric potential is V_i .

Problem 10.16. Capacitor with dielectric

- Modify the Laplace Equation model to include a dielectric medium. That is, create an array of dielectric susceptibilities and implement (10.23) using a relaxation algorithm. Be sure to set the dielectric array elements to unity in free space and inside conductors.
- Test your algorithm by creating a capacitor consisting of $+10$ and -10 potential plates near the center of the grid. Initialize the dielectric susceptibility to two in half the capacitor and run the simulation. Use a Scalar Field element to display the electric potential, but note that some representations of the scalar field are more appropriate than others. Compare the spacing between the contour lines inside and outside the dielectric. Why does the spacing change?
- The bound charge on the surface of a dielectric can be computed by subtracting $V(x, y)$ from the average of the potential at the four nearest neighbor sites. You are, in effect, using (10.17) to solve for the charge. Implement this computation and describe the bound charge on the surface of the dielectric. ■

10.6 Random Walk Solution of Laplace's Equation

10.7 *Fields Due to Moving Charges

10.8 *Maxwell's Equations

10.9 Projects

Part of the difficulty in understanding electromagnetic phenomena is visualizing its three-dimensional nature. Many interesting problems can be posed based on the simple, but nontrivial question of how three-dimensional electromagnetic fields can best be represented visually in various contexts (cf. Belcher and Olbert). However, we have not suggested projects in this area because of their difficulty.

Many of the techniques used in this chapter, for example, the random walk method and the relaxation method for solving Laplace's equation, have applications in other fields, especially problems in fluid flow and transport. Similarly, the multigrid method, discussed in Project 10.17, has far reaching applications.

Project 10.17. Multigrid method

In general, the relaxation method for solving Laplace's equation is very slow even using overrelaxation. The reason is that the local updates of the relaxation method cannot quickly take into account effects at very large length scales. The *multigrid method* greatly improves performance by using relaxation at many length scales. The important idea is to use a relaxation method to find the values of the potential on coarser and coarser grids, and then use the coarse grid values to determine the fine grid values. The fine grid relaxation updates take into account effects at short length scales. If we define the initial grid by a lattice spacing $b = 1$, then the coarser grids are characterized by $b = 2^n$, where n determines the coarseness of the grid and is known as the grid level. We need to decide how to use the fine grid values of the potential to assign values to a coarser grid, and then how to use a coarse grid to assign values to a finer grid. The first step is called *prolongation* and the second step is called *restriction*. There is some flexibility on how to do these two operations. We discuss one approach.

We define the centers of the sites of the coarse grid to be located at the centers of every other site of the fine grid. That is, if the set $\{i, j\}$ represents the positions of the sites of the fine grid, then $\{2i, 2j\}$ represents the positions of the coarse grid sites. The fine grid sites that are at the same position as a coarse grid point are assigned the value of the potential of the corresponding coarse grid point. The fine grid sites that have two coarse grid points as nearest neighbors are assigned the average value of these two coarse grid sites. The other fine grid sites have four coarse grid sites as next nearest neighbors and are assigned the average value of these four coarse grid sites. This prescription specifies how values on the fine grid are computed using the values on the coarse grid.

In the full weighting prolongation method, each coarse grid site receives one fourth of the potential of the fine grid site at the same position, one eighth of the potential for the four nearest neighbor sites of the fine grid, and one sixteenth of the potential for the four next nearest neighbor

points of the fine grid. The sum of these fractions, $1/4 + 4(1/8) + 4(1/16)$, adds up to unity. An alternative procedure, known as half weighting, ignores the next nearest neighbors and uses one half of the potential of the fine grid site at the same position as the coarse grid site.

- a. Create a model that implements the multigrid method using Gauss-Seidel relaxation on a checkerboard lattice (see Problem 10.11b). In its simplest form the model should allow the user to intervene and decide whether to go to a finer or coarser grid, or to remain at the same level for the next relaxation step. Have the simulation print the potential at each site of the current level after each relaxation step. Test your model on a 4×4 grid whose boundary sites are all equal to unity, and whose initial internal sites are set to zero. Make sure that the boundary sites of the coarser grids also are set to unity.
- b. The exact solution for part (a) gives a potential of unity at each point. How many relaxation steps does it take to reach unity within 0.1% at every site by simply using the 4×4 grid? How many steps does it take if you use one coarse grid and continue until the coarse grid values are within 0.1% of unity? Is it necessary to carry out any fine grid relaxation steps to reach the desired accuracy on the fine grid? Next start with the coarsest scale, which is just one site. How many relaxation steps does it take now?
- c. Repeat part (b), but change the boundary so that one side of the boundary is held at a potential of 0.5. Experiment with different sequences of prolongation, restriction, and relaxation.
- d. Assume that the boundary points alternate between zero and unity, and repeat part (b). Does the multigrid method work? Should one go up and down in levels many times instead of staying at the coarsest level and then going down to the finest level?

10.10 Simulations

The following models are implemented in *EJS* and are downloadable from the *OSP* Collection in the ComPADRE digital library.

Point Charge Electric Field

The Point Charge Electric Field model draws a grid of vectors to show the electric field from multiple point charges. The model stores ten particle locations but only the first n particles are visible and only these active particles are used to compute the field. Whenever a charge is added or moved, the electric field is recomputed using a fixed relation. See Section 10.2.

Electric Field Lines

The Electric Field Lines model draws the electric field lines in the vicinity of point charges. The field line is drawn by solving an ODE in response to a mouse click. See Section 10.3.

Laplace Equation

The Laplace Equation model uses the relaxation method to solve the boundary value problem of finding the potential $V(x,y)$ in a charge-free region bounded by conductors. For teaching purposes, the problem is solved using the Jacobi method but the more efficient Gauss-Seidel method is also implemented on a disabled evolution page. See Section 10.5.

Vector Field

The Vector Field model displays the vector field $\mathbf{A} = (A_x, A_y)$. The vector field visualization uses arrows that have the same length but different colors. Using color rather than arrow length to represent a vector field's strength produces a more effective representation of magnitude over a wider range of values. See Appendix A.

Analytic Vector Field

The Analytic Vector Field model. This model is similar to the Vector Field model except that it uses an Analytic Vector Field rather than a Vector Field element to display the field. The analytic element is simpler to use because it accepts two strings, rather than two arrays, representing as input and parses these strings into mathematical expressions to compute the field components. See Appendix A.

Scalar Field Gradient

The Scalar Field Gradient model displays the gradient by overlaying a vector field on scalar field $U(x,y)$. The gradient of a scalar field $U(x,y)$ is a vector field that points in the direction of the greatest rate of change of the scalar field. The Scalar Field Gradient Model computes this gradient using the central difference approximation to the partial derivatives. See Appendix A.

Appendix A: Visualizing Vector Fields

The *Vector Field 2D* and *Analytic Vector Field 2D* elements are similar to their scalar field counterparts except that two components are needed to specify the field. The Vector Field element uses two $n \times m$ arrays to store components whereas the Analytic Vector Field element parses two strings representing analytic functions.

The Vector Field model in this chapter's source code directory uses two interface elements to input $A_x(x,y)$ and $A_y(x,y)$ functions, evaluates these functions on a grid, and displays the vector field $\mathbf{A} = (A_x, A_y)$. The evaluation is done in the fixed relation shown in Listing 10.4.

Listing 10.4: The fixed relation computes the vector field at each grid point.

```
//scalar field is computed at each grid point
for(int i=0; i<m; i++) {                                // loop over grid x
    double xi= _view.vectorField.indexToX(i);           // x-coordinate of grid point
    for(int j=0; j<m; j++) {                             // loop over grid y
        double yj= _view.vectorField.indexToY(j);       // y-coordinate of grid point
```

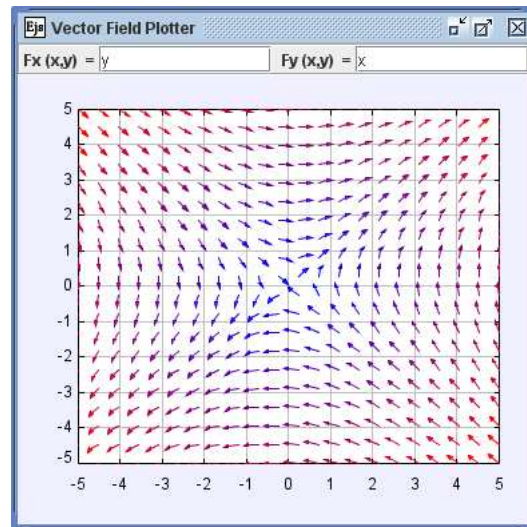


Figure 10.4: Fixed-length color-coded arrows are used to represent a vector field with a wide dynamic range. Pale blue and dark red represent small and large magnitude fields, respectively. These colors are converted to grayscale in black and white reproductions.

```

    xComponentData[i][j]=_view.xFunction.evaluate(xi,yj); // x-component value
    yComponentData[i][j]=_view.yFunction.evaluate(xi,yj); // y-component value
  } // end of y loop
} // end of x loop

```

Run the model and note that the arrows have the same length but different colors. These properties are set in the vector field's inspector. The arrow length property is given in world units so we divide range by the number of grid points. Remove the Length property input in the inspector and run the model again. The arrows now overlap because the default arrow length is the vector magnitude.

Using color rather than arrow length to represent a vector field's strength produces a more effective representation of magnitude over a wider range of values. The color palette is set using the Color Mode property in the Graphics Aspect section of the inspector. The default palette changes the arrow color from the Min Color (default is blue) toward Max Color (default is red) as the magnitude increases. Use the inspector to change the color mode to **GRAYSCALE** and run the model to see how this palette assigns the colors.

The appearance section of the inspector allows us to further control the mapping of field magnitude to color. The Color Mode selects a color pallet and the default Appearance properties set the first color to minimum magnitude vectors and the last color to maximum magnitude arrows. We can, however, assign our own map. For example, setting the Color Component property to 1 uses the color that correspond to 1 for all arrows. Run the model and observe that all vectors are red because the default maximum value is 1 and red is the last color in the default palette. Set the Color Component to 0 and the first color (blue) in the default palette is used. To control the color of every grid point, the Color Component property accepts a `double[][]` array that maps

each `double` value to an arrow color at the corresponding grid point.

The Analytic Vector Field model is slightly different from the Vector Field model because it does not create its own arrays. The analytic vector field automatically creates arrays for us if we provide the number of x and y grid points. Open this model and examine the `X` and `Y` Component properties. The vector component properties are strings that are set to `%xFunction%` and `%yFunction%`, respectively. Percent signs are required to access the global `%xFunction%` and `%yFunction%` string variables. The percent sign is necessary because the property inspector expects a string value and would interpret string input as the actual (literal) value of the property rather than as the name of a model variable.

Because user interface elements are rendered in the order that they appear within a drawing panel in the Tree of Elements, we can overlay a vector field on a scalar field. The Scalar Field Gradient model uses this technique to display the gradient of a scalar field. Examine this model and note that the gradient computation in the fixed relation uses the grid point separation Δx to compute the numerical derivatives using a central difference approximation. We do not compute the derivative at the edge in order to avoid an array out of bounds error.

Exercise 10.18. Divergence

Modify the Vector Field model so as to compute and display the vector field's divergence. ■

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