Chapter 4

Oscillations

15 August 2009

Adapted from *An Introduction to Computer Simulation Methods* by Harvey Gould, Jan Tobochnik, and Wolfgang Christian

We explore the behavior of oscillatory systems, including the simple harmonic oscillator, a simple pendulum, and electrical circuits and we introduce the concept of phase space. We also show how the *EJS* ODE editor is used to solve arrays of differential equations.

4.1 Simple Harmonic Motion

There are many physical systems that undergo regular, repeating motion. Motion that repeats itself at definite intervals, for example, the motion of the Earth about the Sun, is said to be *periodic*. If an object undergoes periodic motion between two limits over the same path, we call the motion *oscillatory*. Examples of oscillatory motion that are familiar to us from our everyday experience include a plucked guitar string and the pendulum in a grandfather clock. Less obvious examples are microscopic phenomena such as the oscillations of the atoms in crystalline solids.

To illustrate the important concepts associated with oscillatory phenomena, consider a block of mass m connected to the free end of a spring. The block slides on a frictionless, horizontal surface (see Figure 4.1). We specify the position of the block by x and take $x = 0$ to be the equilibrium position of the block, that is, the position when the spring is relaxed. If the block is moved from $x = 0$ and then released, the block oscillates along a horizontal line. If the spring is not compressed or stretched too far from $x = 0$, the force on the block at position x is proportional to x:

$$F = -kx. \tag{4.1}$$

The force constant k is a measure of the stiffness of the spring. The negative sign in (4.1) implies that the force acts to restore the block to its equilibrium position. Newton's equation of motion

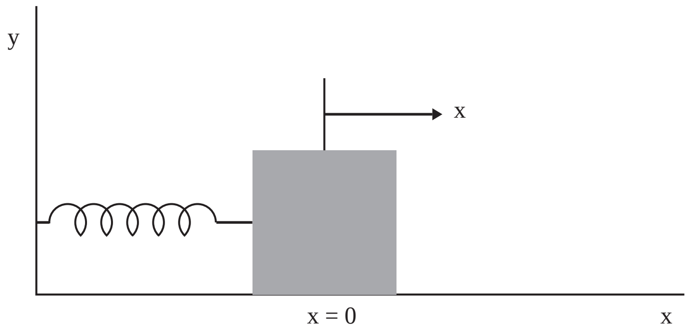

Figure 4.1: A one-dimensional harmonic oscillator. The block slides horizontally on the frictionless surface.

for the block can be written as

$$\frac{d^2x}{dt^2} = -{\omega_0}^2 x, \tag{4.2}$$

where the angular frequency ω_0 is defined by

$${\omega_0}^2 = \frac{k}{m}. \tag{4.3}$$

The dynamical behavior described by (4.2) is called *simple harmonic motion* and can be solved analytically in terms of sine and cosine functions. Because the form of the solution will help us introduce some of the terminology needed to discuss oscillatory motion, we include the solution here. One form of the solution is

$$x(t) = A\cos(\omega_0 t + \delta), \tag{4.4}$$

where A and δ are constants and the argument of the cosine is in radians. It is straightforward to check by substitution that (4.4) is a solution of (4.2). The constants A and δ are called the amplitude and the phase respectively, and are determined by the initial conditions for x and the velocity $v = dx/dt$.

Because the cosine is a periodic function with period 2π, we know that $x(t)$ in (4.4) also is periodic. We define the period T as the smallest time for which the motion repeats itself, that is,

$$x(t + T) = x(t). \tag{4.5}$$

Because $\omega_0 T$ corresponds to one cycle, we have

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}}. \tag{4.6}$$

The frequency ν of the motion is the number of cycles per second and is given by $\nu = 1/T$. Note that T depends on the ratio k/m and not on A and δ. Hence the period of simple harmonic motion is independent of the amplitude of the motion.

Although the position and velocity of the oscillator are continuously changing, the total energy E remains constant and is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2. \tag{4.7}$$

The two terms in (4.7) are the kinetic and potential energies, respectively.

We introduced the SHO model in Chapter 3 and now study it in more detail. Open this chapter's SHO Solver Comparison model in the Chapter 4 source directory and use it as a template for the following problems.

Problem 4.1. Energy conservation

a. Add an energy error (energy drift) plot $\Delta E = E_n - E_0$ to the SHO Solver Comparison model where E_n is the energy after the nth evolution step and E_0 is the initial energy.

b. Describe the qualitative differences of the SHO energy drift using the various algorithms in the SHO Comparison model. Which algorithm is most consistent with the requirement of conservation of energy? For fixed Δt, which algorithm yields better results for the position in comparison to the analytical solution (4.4)? Is the requirement of conservation of energy consistent with the relative accuracy of the computed positions?

c. Choose an algorithm based on your criteria, and determine the values of Δt that are needed to conserve the total energy to within 0.1% over one cycle for $\omega_0 = 3$ and for $\omega_0 = 12$. Can you use the same value of Δt for both values of ω_0? If not, how do the values of Δt correspond to the relative values of the period in the two cases?

Problem 4.2. Analysis of simple harmonic motion

a. Use your results from Problem 4.1 to select an appropriate numerical algorithm and value of Δt for the simple harmonic oscillator and modify your model so that the time dependence of the potential and kinetic energies is plotted. Where in the cycle is the kinetic energy (potential energy) a maximum?

b. Numerically compute the average value of the kinetic energy and the potential energy during a complete cycle. What is the relation between the two averages?

c. Compute $x(t)$ for different values of A and show that the shape of $x(t)$ is independent of A, that is, show that $x(t)/A$ is a *universal* function of t for a fixed value of ω_0. In what units should the time be measured so that the ratio $x(t)/A$ is independent of ω_0?

d. The dynamical behavior of the one-dimensional oscillator is completely specified by $x(t)$ and $p(t)$, where p is the momentum of the oscillator. These quantities may be interpreted as the coordinates of a point in a two-dimensional space known as *phase space*. As the time increases, the point $(x(t), p(t))$ moves along a trajectory in phase space. Modify your model so that the momentum p is plotted as a function of x, that is, choose p and x as the vertical and horizontal axes respectively. Choose $\omega_0 = 3$ and compute the phase space trajectory for the initial condition $x(t = 0) = 1, v(t = 0) = 0$. What is the shape of this trajectory? What is the shape for the initial conditions, $x(t=0)=0, v(t=0)=1$ and $x(t=0)=4, v(t=0)=0$? Do you find a different phase trajectory for each initial condition? What physical quantity distinguishes the phase trajectories? Is the motion of a representative point (x,p) always in the clockwise or counterclockwise direction?

Problem 4.3. Lissajous figures

A computer display can be used to simulate the output seen on an oscilloscope. Imagine that the vertical and horizontal inputs to an oscilloscope are sinusoidal in time, that is, $x = A_x \sin(\omega_x t + \phi_x)$ and $y = A_y \sin(\omega_y t + \phi_y)$. If the curve that is drawn repeats itself, such a curve is called a *Lissajous figure*. Create a model to display y versus x, as t advances from $t = 0$. First choose $A_x = A_y = 1$, $\omega_x = 2$, $\omega_y = 3$, $\phi_x = \pi/6$, and $\phi_y = \pi/4$. For what values of the angular frequencies ω_x and ω_y do you obtain a Lissajous figure? How do the phase factors ϕ_x and ϕ_y and the amplitudes A_x and A_y affect the curves?

The time dependent models we have developed generate time-series data as variables evolve but there are many phenomena that evolve entire functions $f(x,t)$. Such *wave functions* are ubiquitous in nature and give rise to important oscillatory phenomena such as beats and standing waves. We will study the behavior of waves more systematically in Chapter **??** but we start to investigate their behavior in Problem 4.4 in order to learn how to plot time-varying functions. The Traveling Wave model in the Chapter 4 source directory plots $A\sin(kx + \omega t)$ from $x = x_{\min}$ to $x = x_{\max}$ as a function of t. For simplicity we take $A = 1$, $\omega = 2\pi$, and $k = 2\pi/\lambda$ with wavelength $\lambda = 2$. The model uses an array to store the wave function and updates the entire wave function after every evolution step.

Problem 4.4. Superposition of waves

a. Add a circle to the model that is draggable in the x-direction but that follows the height of the wave in the y-direction. *Hint:* Use the circle's On Drag action to set its y-position from the wave function.

b. Modify the Traveling Wave model so that it plots the sum of $y_1 = \sin(kx - \omega t)$ and $y_2 = \sin(kx + \omega t)$. The quantity $y_1 + y_2$ corresponds to the superposition of two waves. Choose $\lambda = 2$ and $\omega = 2\pi$. What kind of a wave do you obtain?

c. Modify your model to demonstrate beats by plotting $y_1 + y_2$ as a function of time in the range $x_{\min} = -10$ and $x_{\max} = 10$. Determine the beat frequency for each of the following superpositions: $y_1(x,t) = \sin[8.4(x - 1.1t)]$, $y_2(x,t) = \sin[8.0(x - 1.1t)]$; $y_1(x,t) = \sin[8.4(x - 1.2t)]$, $y_2(x,t) = \sin[8.0(x - 1.0t)]$; and $y_1(x,t) = \sin[8.4(x - 1.0t)]$, $y_2(x,t) = \sin[8.0(x - 1.2t)]$. What difference do you observe between these superpositions?

4.2 The Motion of a Pendulum

A common example of a mechanical system that exhibits oscillatory motion is the simple pendulum (see Figure 4.2). A simple pendulum is an idealized system consisting of a particle or bob of mass m attached to the lower end of a rigid rod of length L and negligible mass; the upper end of the rod

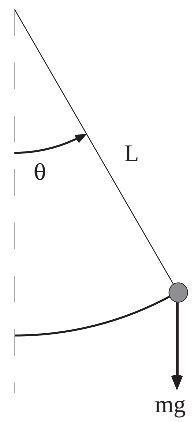

Figure 4.2: Force diagram for a simple pendulum. The angle θ is measured from the vertical direction and is positive if the mass is to the right of the vertical and negative if it is to the left.

pivots without friction. If the bob is pulled to one side from its equilibrium position and released, the pendulum swings in a vertical plane.

Because the bob is constrained to move along the arc of a circle of radius L about the center O, the bob's position is specified by its arc length or by the angle θ (see Figure 4.2). The linear (tangential) velocity and acceleration of the bob as measured along the arc are given by

$$v = L\frac{d\theta}{dt} \tag{4.8}$$

$$a = L\frac{d^2\theta}{dt^2}. \tag{4.9}$$

In the absence of friction, two forces act on the bob: the force mg vertically downward and the force of the rod which is directed inward to the center if $|\theta| < \pi/2$. Note that the effect of the rigid rod is to constrain the motion of the bob along the arc. From Figure 4.2, we can see that the component of mg along the arc is $mg\sin\theta$ in the direction of decreasing θ. Hence, the equation of motion can be written as

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta, \tag{4.10}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta. \tag{4.11}$$

Equation (4.11) is an example of a nonlinear equation because $\sin\theta$ rather than θ appears. Most nonlinear equations do not have analytical solutions in terms of well-known functions, and (4.11) is no exception. However, if the amplitude of the pendulum oscillations is sufficiently small, then $\sin\theta \approx \theta$, and (4.11) reduces to

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta. \qquad (\theta \ll 1) \tag{4.12}$$

Remember that θ is measured in radians.

Part of the fun of studying physics comes from realizing that equations that appear in different contexts often are similar. An example can be seen by comparing (4.2) and (4.12). If we associate x with θ, we see that the two equations are identical in form, and we can immediately conclude that for $\theta \ll 1$, the period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}. \qquad \text{(small amplitude oscillations)} \tag{4.13}$$

One way to understand the motion of a pendulum with large oscillations is to solve (4.11) numerically. Because we know that the numerical solutions must be consistent with conservation of energy, we derive the form of the total energy here. The potential energy can be found from the following considerations. If the rod is deflected by the angle θ, then the bob is raised by the distance $h = L - L\cos\theta$ (see Figure 4.2). Hence, the potential energy of the bob in the gravitational field of the earth is

$$U = mgh = mgL(1 - \cos\theta), \tag{4.14}$$

where the zero of the potential energy corresponds to $\theta = 0$. Because the kinetic energy of the pendulum is $\frac{1}{2}mv^2 = \frac{1}{2}mL^2(d\theta/dt)^2$, the total energy E of the pendulum is

$$E = \frac{1}{2}mL^2\Big(\frac{d\theta}{dt}\Big)^2 + mgL(1 - \cos\theta). \tag{4.15}$$

Problem 4.5. Oscillations of a pendulum

a. Modify the Simple Pendulum model in the Chapter 4 source directory so that the numerical solution is plotted. Note that this simulation uses units such that $g = 1$ to slow the animation. Note also that the plotting panel's Axis Type property has been set to display polar coordinates and that the angle θ rotates multiple elements within the view as explained in Appendix 4A.

b. Make the necessary changes to the Simple Pendulum model so that the analytical solution for small angles also is plotted. Test the model at sufficiently small amplitudes so that $\sin\theta \approx \theta$. Choose the initial conditions $\theta(t = 0) = 0.2$ and $d\theta(t = 0)/dt = 0$. Determine the period numerically and compare your result to the expected analytical result for small amplitudes $\omega_0 = \sqrt{g/L}$. Explain your method for determining the period. Estimate the error due to the small angle approximation for these initial conditions.

c. Consider larger amplitudes and make plots of $\theta(t)$ and $d\theta(t)/dt$ versus t for the initial conditions $\theta(t = 0) = 0.1$, 0.2, 0.4, 0.8, and 1.0 with $d\theta(t = 0)/dt = 0$. Choose Δt so that the numerical algorithm generates a stable solution, that is, monitor the total energy and ensure that it does not drift from its initial value. Describe the qualitative behavior of θ and $d\theta/dt$. What is the period T and the amplitude $\theta_{\max}$ in each case? Plot T versus $\theta_{\max}$ and discuss the qualitative dependence of the period on the amplitude. How do your results for T compare in the linear and nonlinear cases, for example, which period is larger? Explain the relative values of T in terms of the relative magnitudes of the restoring force in the two cases.

4.3 Damped Harmonic Oscillator

We know from experience that most oscillatory motion in nature gradually decreases until the displacement becomes zero; such motion is said to be *damped* and the system is said to be *dissipative* rather than conservative. As an example of a damped harmonic oscillator, consider the motion of the block in Figure 4.1 when a horizontal drag force is included. For small velocities, it is a reasonable approximation to assume that the drag force is proportional to the first power of the velocity. In this case the equation of motion can be written as

$$\frac{d^2x}{dt^2} = -{\omega_0}^2 x - \gamma \frac{dx}{dt}. \tag{4.16}$$

The *damping coefficient* γ is a measure of the magnitude of the drag term. Note that the drag force in (4.16) opposes the motion. We simulate the behavior of the damped linear oscillator in Problem 4.6.

Problem 4.6. Damped linear oscillator

a. Incorporate the effects of damping into your Simple Harmonic Oscillator model and plot the time dependence of the position and the velocity. Describe the qualitative behavior of $x(t)$ and $v(t)$ for $\omega_0 = 3$ and $\gamma = 0.5$ with $x(t = 0) = 1$, $v(t = 0) = 0$.

b. The period of the motion is the time between successive maxima of $x(t)$. Use an ODE event to measure the period and corresponding angular frequency and compare their values to the undamped case. Is the period longer or shorter? Make additional runs for $\gamma = 1$, 2, and 3. Does the period increase or decrease with greater damping? Why?

c. The amplitude is the maximum value of x during one cycle. Compute the *relaxation time* τ, the time it takes for the amplitude of an oscillation to decrease by $1/e \approx 0.37$ from its maximum value. Is the value of τ constant throughout the motion? Compute τ for the values of γ considered in part (b) and discuss the qualitative dependence of τ on γ.

d. Plot the total energy as a function of time for the values of γ considered in part (b). If the decrease in energy is not monotonic, explain.

e. Compute the time dependence of $x(t)$ and $v(t)$ for $\gamma = 4$, 5, 6, 7, and 8. Is the motion oscillatory for all γ? How can you characterize the decay? For fixed ω_0, the oscillator is said to be *critically damped* at the smallest value of γ for which the decay to equilibrium is monotonic. For what value of γ does critical damping occur for $\omega_0 = 4$ and $\omega_0 = 2$? For each value of ω_0, compute the value of γ for which the system approaches equilibrium most quickly.

f. Compute the phase space diagram for $\omega_0 = 3$ and $\gamma = 0.5$, 2, 4, 6, and 8. Why does the phase space trajectory converge to the origin, $x = 0$, $v = 0$? This point is called an *attractor*. Are these qualitative features of the phase space plot independent of γ?

Problem 4.7. Damped nonlinear pendulum

Consider a damped pendulum with $\omega_0 = \sqrt{g/L}$ and a damping term equal to $-\gamma d\theta/dt$. Choose $\gamma = 1$ and the initial condition $\theta(t = 0) = 0.2, d\theta(t = 0)/dt = 0$. In what ways is the motion of the damped nonlinear pendulum similar to the damped linear oscillator? In what ways is it different? What is the shape of the phase space trajectory for the initial condition $\theta(t=0)=1, \omega(t=0)=0$? Do you find a different phase trajectory for other initial conditions? Remember that θ is restricted to be between $-\pi$ and $+\pi$.

4.4 Response to External Forces

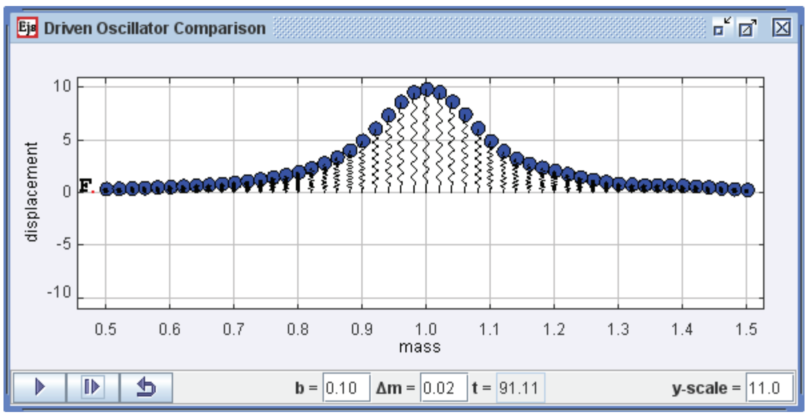

Figure 4.3: A comparison of fifty-one oscillators with differing natural frequencies. All oscillators are driven with a synchronous external sinusoidal force.

How can we determine the period of an oscillator that is not already in motion? The obvious way is to disturb the system, for example, to displace the mass and observe its motion. We will find that the nature of the response of the system to a small perturbation tells us something about the nature of the system in the absence of the perturbation.

Consider the driven damped linear oscillator with an external force $F(t)$ in addition to the linear restoring force and linear damping force. The equation of motion can be written as

$$\frac{d^2x}{dt^2} = -{\omega_0}^2 x - \gamma v + \frac{1}{m}F(t). \tag{4.17}$$

It is customary to interpret the response of the system in terms of the displacement x rather than the velocity v.

The time dependence of $F(t)$ in (4.17) is arbitrary. Because many forces in nature are periodic, we first consider the form

$$\frac{1}{m}F(t) = A_0 \cos\omega t, \tag{4.18}$$

where ω is the angular frequency of the driving force.

The Driven SHO Comparison model shown in Figure 4.3 and available in the Chapter 4 source directory displays fifty-one driven harmonic oscillators with identical spring constants but slightly different masses. These oscillators are driven with a sinusoidal force with an angular frequency equal to $(k/m)^{1/2}$ for the oscillator at the center of the display. Run this model and observe the differing response of each of theses oscillators. We study these differences in Problems 4.8 and 4.9.

Exercise 4.8. Driven damped linear oscillator comparison

Run the Driven SHO Comparison model and note the red arrow to the left of the oscillators showing the driving force that is applied to each oscillator.

a. The driven damped linear oscillators shown settle into a steady state after a certain time. How long does it take for the steady state behavior to emerge if the damping coefficient is $b = 0.1$? If $b = 0.2$?

b. How does the damping coefficient affect the maximum displacement of the center mass?

c. Run the model and describe the steady state pattern at the instant when the driving force is zero? When the force is a maximum? Why are these two patterns different?

Problem 4.9. Response of a driven damped linear oscillator analysis

a. Modify your Simple Harmonic Oscillator model so that an external force of the form (4.18) is included. Add this force by creating a custom method so that the rate in the ODE evolution page is written as $dv_x/dt = force(x, v_x, t)/m$. The angular frequency of the driving force should be added as a global variable with an input field.

b. Choose $\omega_0 = 3$, $\gamma = 0.5$, $\omega = 2$ and the amplitude of the external force $A_0 = 1$ for all runs unless otherwise stated. For these values of ω_0 and γ, the dynamical behavior in the absence of an external force corresponds to an underdamped oscillator. Plot $x(t)$ versus t in the presence of the external force with the initial condition, $x(t = 0) = 1, v(t = 0) = 0$. How does the qualitative behavior of $x(t)$ differ from the nonperturbed case? What is the period and angular frequency of $x(t)$ after several oscillations? Repeat the same observations for $x(t)$ with $x(t = 0) = 0, v(t = 0) = 1$. Identify a transient part of $x(t)$ that depends on the initial conditions and decays in time, and a steady state part that dominates at longer times and is independent of the initial conditions.

c. Compute $x(t)$ for several combinations of ω_0 and ω. What is the period and angular frequency of the steady state motion in each case? What parameters determine the frequency of the steady state behavior?

d. A measure of the long-term behavior of the driven harmonic oscillator is the amplitude of the steady state displacement $A(\omega)$, which can be be computed for a given value of ω if the simulation is run until a steady state has been achieved. One way to determine A is to use a zero crossing event to record the time and position when the velocity is zero.

e. Measure the amplitude and phase shift to verify that the steady state behavior of $x(t)$ is given by

$$x(t) = A(\omega)\cos(\omega t + \delta). \tag{4.19}$$

The quantity δ is the phase difference between the applied force and the steady state motion. Compute $A(\omega)$ and $\delta(\omega)$ for $\omega_0 = 3$, $\gamma = 0.5$, and $\omega = 0$, 1.0, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, and 3.4. Choose the initial condition, $x(t=0)=0, v(t=0)=0$. Repeat the simulation for $\gamma = 3.0$, and plot $A(\omega)$ and $\delta(\omega)$ versus ω for the two values of γ. Discuss the qualitative behavior of $A(\omega)$ and $\delta(\omega)$ for the two values of γ. If $A(\omega)$ has a maximum, determine the angular frequency $\omega_{\rm max}$ at which the maximum of A occurs. Is the value of $\omega_{\rm max}$ close to the natural angular frequency ω_0? Compare $\omega_{\rm max}$ to ω_0 and to the frequency of the damped linear oscillator in the absence of an external force.

f. Compute $x(t)$ and $A(\omega)$ for a damped linear oscillator with the amplitude of the external force $A_0 = 4$. How do the steady state results for $x(t)$ and $A(\omega)$ compare to the case $A_0 = 1$? Does the transient behavior of $x(t)$ satisfy the same relation as the steady state behavior?

g. What is the shape of the phase space trajectory for the initial condition $x(t=0)=1, v(t=0)=0$? Do you find a different phase trajectory for other initial conditions?

h. Why is $A(\omega=0) < A(\omega)$ for small ω? Why does $A(\omega) \to 0$ for $\omega \gg \omega_0$?

i. Does the mean kinetic energy resonate at the same frequency as does the amplitude? Compute the mean kinetic energy over one cycle once steady state conditions have been reached. Choose $\omega_0 = 3$ and $\gamma = 0.5$.

In Problem 4.9 we found that the response of the damped harmonic oscillator to an external driving force is linear. For example, if the magnitude of the external force is doubled, then the magnitude of the steady state motion also is doubled. This behavior is a consequence of the linear nature of the equation of motion. When a particle is subject to nonlinear forces, the response can be much more complicated (see Section 4.2 and Chapter 6).

For many problems, the sinusoidal driving force in (4.18) is not realistic. Another example of an external force can be found by observing someone pushing a child on a swing. Because the force is nonzero only for short intervals of time, this type of force is impulsive. In the following problem, we consider the response of a damped linear oscillator to an impulsive force.

*__Problem 4.10.__ Response of a damped linear oscillator to nonsinusoidal external forces

a. Assume a swing can be modeled by a damped linear oscillator and that the swing is pushed at the instant it reaches its maximum negative displacement. The effect of this push (impulse) is to change the velocity and this push occurs once every period. One way to implement this scenario is with an ODE event. Add a positive crossing ODE event to your undriven SHO model with the following zero condition:

```
return v;
```

This event is triggered when the velocity changes from positive to negative. Because we assume the push is instantaneous, we model the effect by changing the velocity (momentum)

```
v += 0.1;
```

Determine the steady state amplitude $A(\omega)$ for ω = 1.0, 1.3, 1.4, 1.5, 1.6, 2.5, 3.0, and 3.5. The corresponding period of the impulse is given by $T = 2\pi/\omega$. Choose $\omega_0 = 3$ and $\gamma = 0.5$. Are your results consistent with your experience of pushing a swing and with the comparable results of Problem 4.9?

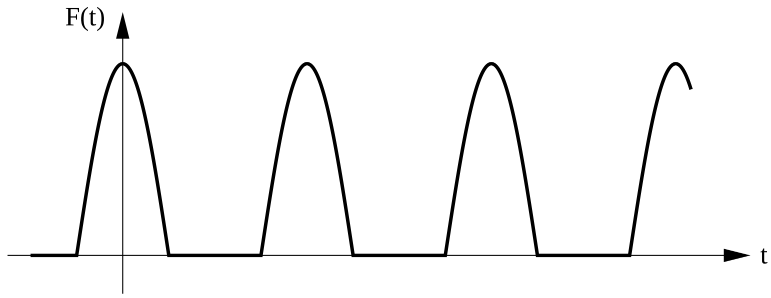

Figure 4.4: A half-wave driving force corresponding to the positive part of a cosine function.

b. Consider the response to a half-wave external force consisting of the positive part of a cosine function (see Figure 4.4). This force can be coded as

```
double force=Math.max(0,Math.cos(omega*t));
```

but we must be careful to choose a small time step so as not to miss the onset of the discontinuous force.[1]

Compute $A(\omega)$ for $\omega_0 = 3$ and $\gamma = 0.5$. At what values of ω does $A(\omega)$ have a relative maxima? Is the half-wave cosine driving force equivalent to a sum of cosine functions of different frequencies? For example, does $A(\omega)$ have more than one resonance?

c. Compute the steady state response $x(t)$ to the external force

$$\frac{1}{m}F(t) = \frac{1}{\pi} + \frac{1}{2}\cos t + \frac{2}{3\pi}\cos 2t - \frac{2}{15\pi}\cos 4t. \tag{4.20}$$

How does a plot of $F(t)$ versus t compare to the half-wave cosine function? Use your results to conjecture a principle of superposition for the solutions to linear equations.

In many of the problems in this chapter we have asked you to draw a phase space plot for a single oscillator. This plot provides a convenient representation of both the position and velocity. When we study chaotic phenomena such plots will become almost indispensable (see Chapter 6). Here we will consider an important feature of phase space trajectories for conservative systems.

If there are no external forces, the undamped simple harmonic oscillator and undamped pendulum are examples of conservative systems, that is, systems for which the total energy is a constant. In Problems 4.11 and 4.12 we will study two general properties of conservative systems, the non-intersecting nature of their trajectories in phase space and the preservation of area in phase space. These concepts will become more important when we study the properties of conservative systems with more than one degree of freedom.

Problem 4.11. Trajectory of a simple harmonic oscillator in phase space

[1]Strictly speaking, finite difference ODE algorithms assume that the rates are continuous but the difference is not noticeable if the time step is small. The correct approach for high-accuracy modeling is trigger an event at the discontinuity that resets the differential equation initial conditions before continuding the solution.

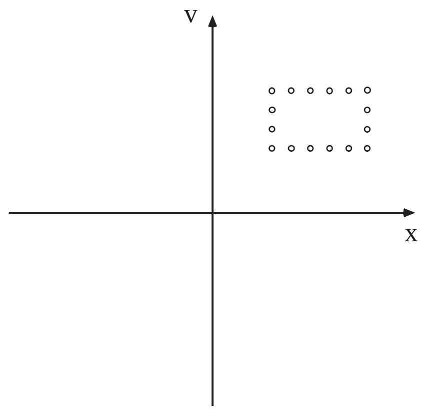

Figure 4.5: What happens to a given area in phase space for conservative systems?

a. We explore the phase space behavior of a single harmonic oscillator by simulating N initial conditions simultaneously. Modify the Driven SHO Comparison model to simulate N identical simple harmonic oscillators each of which is represented by a small circle centered at its position and velocity in phase space as shown in Fig. 4.5. Choose $N = 16$ and consider random initial positions and velocities. Do the phase space trajectories for different initial conditions ever cross? Explain your answer in terms of the uniqueness of trajectories in a deterministic system.

b. Choose a set of initial conditions that form a rectangle (see Figure 4.5). Does the shape of this area change with time? What happens to the total area in comparison to the original area?

Problem 4.12. Trajectory of a pendulum in phase space

a. Modify your model from Problem 4.11 so that the phase space trajectories (ω versus θ) of $N = 16$ pendula with different initial conditions can be compared. Plot several phase space trajectories for different values of the total energy. Are the phase space trajectories closed? Does the shape of the trajectory depend on the total energy?

b. Choose a set of initial conditions that form a rectangle in phase space, and plot the state of each pendulum as a circle. Does the shape of this area change with time? What happens to the total area?

4.5 Frequency Response

Measuring the frequency response of an oscillator by adjusting the drive frequency and observing the steady-state behavior is tedious. The SHO Frequency Response model shown in Figure 4.6 automates this analysis by performing evolution steps with Δt equal to 1000 drive cycles. In order to obtain an accurate solution for this long time interval, we must advance the dynamical variable using many smaller time steps. Unfortunately, we do not know the optimal step size nor do we wish to track of the many steps needed. *EJS* elegantly solves both problems using *adaptive* step

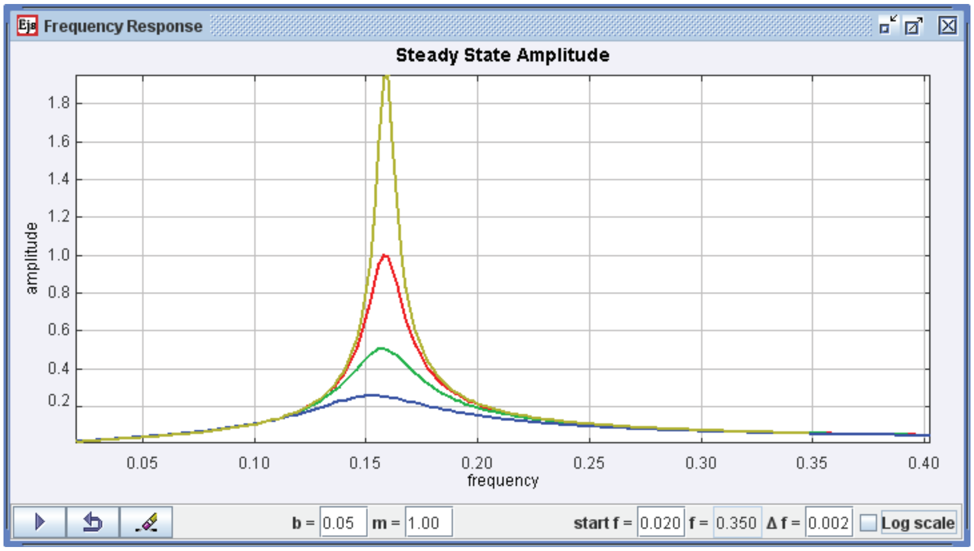

Figure 4.6: .

algorithms and by performing multiple ODE steps to achieve the requested evolution. Adaptive step algorithms, such as Cash-Karp and Dormand-Prince, automatically adjust the ODE step size so as to achieve a specified accuracy. This optimal step size almost never matches the evolution step specified by the model. If the optimal step size is larger than the evolution step, *EJS* computes the solution using the optimal step size and interpolates this solution back in time to set the dynamical variables. If the optimal step size is smaller than the requested step, *EJS* performs multiple steps until the requested time is achieved. This hybrid approach produces a solution that is uniformly spaced and is guaranteed to be as accurate as the requested tolerance. Because the ODE algorithm operates at high efficiency, the steady state can be computed within a single evolution step in the SHO Frequency Response model. (See Appendix 5.1A for a further discussion of adaptive solvers.)

We study the SHO Frequency Response model in Problem 4.13 to understand the physics and to learn how to adapt the ODE step size. The ODE solver advances the dynamical variables using the Cash-Karp adaptive step algorithm to advance the differential equation for many cycles with a solution tolerance of 10^{-4}. The Record Data workpanel then computes the amplitude of oscillation and increments the frequency in preparation for the next evolution step. Note that the input field actions for the mass and damping coefficient initialize the ODE but do not clear the display so that multiple resonance curves can be compared.

Problem 4.13. Frequency response

a. How does the resonance change if mass is increased while the damping remains constant?

b. How does the resonance change if the damping parameter is increased while the mass remains constant?

c. Vary the number of transient cycles to estimate how many drive cycles can be computed without slowing the simulation.

d. Modify the model to simulate pendulum motion. How does the shape of the pendulum resonance differ from the simple harmonic oscillator resonance? How do these differences depend on the damping parameter?

4.6 Electrical Circuit Oscillations

In this section we discuss several electrical analogues of the mechanical systems that we have considered. Although the equations of motion are similar in form, it is convenient to consider electrical circuits separately, because the nature of the questions of interest is somewhat different.

The starting point for electrical circuit theory is Kirchhoff's loop rule, which states that the sum of the voltage drops around a closed path of an electrical circuit is zero. This law is a consequence of conservation of energy, because a voltage drop represents the amount of energy that is lost or gained when a unit charge passes through a circuit element. The relations for the voltage drops across each circuit element are summarized in Table 4.1.

element	voltage drop	symbol	units
resistor	$V_\mathrm{R} = IR$	resistance R	ohms (Ω)
capacitor	$V_\mathrm{C} = Q/C$	capacitance C	farads (F)
inductor	$V_\mathrm{L} = L\,dI/dt$	inductance L	henries (H)

Table 4.1: The voltage drops across the basic electrical circuit elements. Q is the charge (coulombs) on one plate of the capacitor, and I is the current (amperes).

Imagine an electrical circuit with an alternating voltage source $V_s(t)$ attached in series to a resistor, inductor, and capacitor (see Figure 4.7). The corresponding loop equation is

$$V_\mathrm{L} + V_\mathrm{R} + V_\mathrm{C} = V_s(t). \tag{4.21}$$

The voltage source term V_s in (4.21) is the *emf* and is measured in units of volts. If we substitute the relationships shown in Table 4.1, we find

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_s(t), \tag{4.22}$$

where we have used the definition of current $I = dQ/dt$. We see that (4.22) for the series RLC circuit is identical in form to the damped harmonic oscillator (4.17). The analogies between ideal electrical circuits and mechanical systems are summarized in Table 4.2.

Although we are already familiar with (4.22), we first consider the dynamical behavior of an RC circuit described by

$$RI(t) = R\frac{dQ}{dt} = V_s(t) - \frac{Q}{C}. \tag{4.23}$$

Two RC circuits corresponding to (4.23) are shown in Figure 4.8. Although the loop equation (4.23) is identical regardless of the order of placement of the capacitor and resistor in Figure 4.8, the output voltage measured by the oscilloscope in Figure 4.8 is different. We will see in Problem 4.14

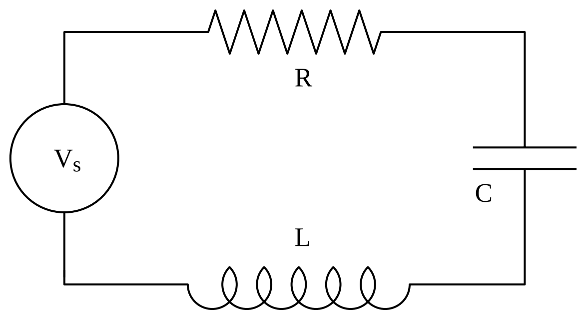

Figure 4.7: A simple series RLC circuit with a voltage source V_s.

Electric circuit	Mechanical system
charge Q	displacement x
current $I = dQ/dt$	velocity $v = dx/dt$
voltage drop	force
inductance L	mass m
inverse capacitance $1/C$	spring constant k
resistance R	damping γ

Table 4.2: Analogies between electrical parameters and mechanical parameters.

that these circuits act as filters that pass voltage components of certain frequencies while rejecting others.

An advantage of a computer simulation of an electrical circuit is that the measurement of a voltage drop across a circuit element does not affect the properties of the circuit. In fact, digital computers often are used to optimize the design of circuits for special applications. The RC Circuit model simulates a resistor and a capacitor in series with either a sinusoidal or a square wave source voltage $V_s(t$ and plots the time dependence of the voltage drops across the source, the resistor, and the capacitor. You are asked to modify this model in Problem 4.14.

Problem 4.14. Simple filter circuits

a. Modify the RC Circuit model to display the voltage across the resistor in order to simulate the voltages in an RC high-pass filter. Your model should plot the voltage across the resistor, $V_{\rm R}$, in addition to the voltage across the source V_s and the voltage across the capacitor $V_{\rm C}$. Run this simulation with $R = 1000\,\Omega$ and $C = 1.0\,\mu$F (10^{-6} farads). Find the steady state amplitude of the voltage drops across the resistor and across the capacitor as a function of the angular frequency ω of the source voltage $V_s = \cos\omega t$. Consider the frequencies $f = 10$, 50, 100, 160, 200, 500, 1000, 5000, and 10000 Hz. (Remember that $\omega = 2\pi f$.) Use these results to explain how an LR circuit can be used as a low pass or a high pass filter. Note that the time step Δt should be no larger than 1/10 of the voltage source period to obtain a smooth curve. What is a reasonable value of Δt for $f = 10000$ Hz? The model uses an adaptive ODE solver so the internal (hidden) time step may be smaller to achieve the specified accuracy.

b. Examine the voltage drops across the inductor and resistor as a function of time and determine

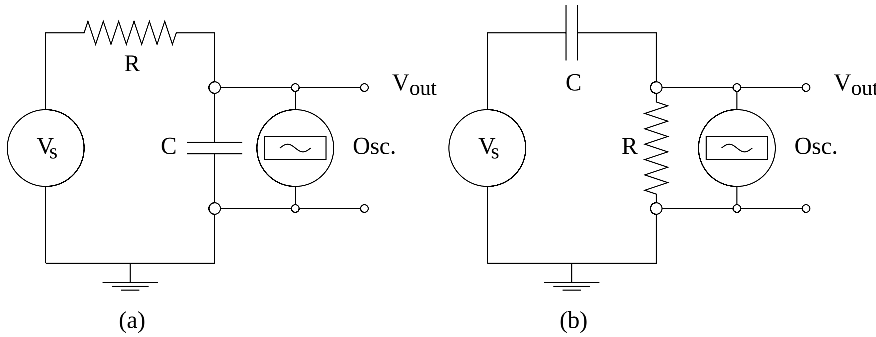

Figure 4.8: Examples of RC circuits used as low and high pass filters. Which circuit is which?

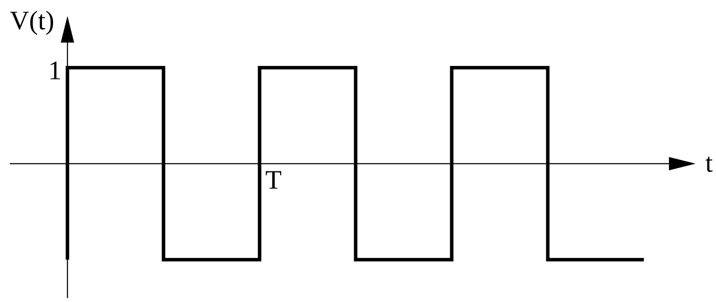

Figure 4.9: Square wave voltage with period T and unit amplitude.

the phase differences ϕ_R and ϕ_L between the resistor and the voltage source and the inductor and the voltage source. The phase difference ϕ can be found by finding the time t_m between the corresponding maxima of the voltages. Because ϕ is usually expressed in radians, we have the relation $\phi/2\pi = t_m/T$, where T is the period of the oscillation. What is the phase difference ϕ_C between the capacitor and the voltage source and the phase difference ϕ_R between the resistor and the voltage source? Do these phase differences depend on ω? Does the current lead or lag the voltage, that is, does the maxima of $V_{\rm R}(t)$ come before or after the maxima of $V_s(t)$? What is the phase difference between the capacitor and the resistor? Does the latter difference depend on ω?

Problem 4.15. RC frequency response

The output voltage depends on where the digital oscilloscope is connected. What is the output voltage of the oscilloscope in Figure 4.8a? Modify the SHO Frequency Response model in Problem 4.13 to display the ratio of the steady state amplitude of the output voltage to the amplitude of the input voltage as a function of ω. Use a logarithmic scale for ω. What range of frequencies is passed? Does this circuit act as a high pass or a low pass filter? Answer the same questions for the oscilloscope in Figure 4.8b. Use your results to explain the operation of a high and low pass filter. Compute the value of the cutoff frequency for which the amplitude of the output voltage drops to $1/\sqrt{2}$ (half-power) of the input value. How is the cutoff frequency related to RC?

Exercise 4.16. Square wave response of an RC circuit

The RC Circuit model has a square wave voltage option as shown in Figure 4.9. Use a $1.0\,\mu$F capacitor and a $3000\,\Omega$ resistor and observe the computed voltage drop across the capacitor as a function of time. Make sure the period of the square wave is long enough so that the capacitor is fully charged during one half-cycle. What is the approximate time dependence of $V_C(t)$ while the capacitor is charging (discharging)?

We now study the steady state behavior of the series RLC circuit shown in Figure 4.7 and represented by (4.22) using the RLC Circuit model in this chapter's source code directory. The response of an electrical circuit is the current rather than the charge on the capacitor. Because we have simulated the analogous mechanical system, we already know much about the behavior of driven the driven RLC model. Nonetheless, we will find several interesting features of AC electrical circuits in the following problems.

Problem 4.17. RLC circuit

a. The RLC Circuit model computes the voltage drops across the resistor, capacitor, and inductor circuit elements with an AC voltage source of the form $V(t) = V_0 \cos \omega t$. Modify the model to display the current I as a function of time and determine the maximum steady state current $I_{\max}$ for different values of ω with $R = 100\,\Omega$, $C = 3.0\,\mu$F, and $L = 2\,$mH. Find the value of ω at which the steady state current amplitude is a maximum. This value of ω is the *resonant frequency*.

b. Observe the time dependence of the voltage drops across each circuit element for approximately fifteen frequencies ranging from 1/10 to 10 times the resonant frequency. Sketch the *resonance curve* by plotting the steady state current amplitude as a function of ω.

c. Vary the resistance and observe the steady state voltage amplitudes across the inductor and the capacitor at the resonant frequency. How do these voltage drops compare to the voltage drop across the resistor and the source voltage? Also compare the relative phases of V_C and V_L at resonance. Explain how an RLC circuit can be used to amplify the input voltage.

Problem 4.18. RLC power

The instantaneous power $P(t)$ is the product of the current through and the voltage across a circuit element. Modify the RLC Circuit model to display the instantaneous power delivered to each circuit element including the voltage source V_s. Explain why the instantaneous power can be positive or negative. Examine your plots to estimate the average power delivered to the various circuit elements at resonance.

Problem 4.19. RLC frequency response

a. Modify the SHO Frequency Response model introduced in Problem 4.13 to display the RLC steady state current as a function of frequency.

b. The sharpness of the resonance curve of an AC circuit is related to the quality factor or Q value. (Q should not be confused with the charge on the capacitor.) The sharper the resonance, the larger the value of Q. Circuits with high Q (and hence a sharp resonance) are useful for tuning circuits in a radio so that only one station is heard at a time. We define $Q = \omega_0/\Delta\omega$, where the width $\Delta\omega$ is the frequency interval between points on the resonance curve $I_{\max}(\omega)$ that are $\sqrt{2}/2$ of $I_{\max}$ at its maximum. Compute Q for the values of R, L, and C given in part (a). Change the value of R by 10% and compute the corresponding percentage change in Q. What is the corresponding change in Q if L or C is changed by 10%?

c. The ratio of the steady state amplitude of the sinusoidal source voltage to the amplitude of the current is called the *impedance* Z of the circuit, that is, $Z = V_{\max}/I_{\max}$. This definition of Z is a generalization of the resistance that is defined by the relation $V = IR$ for direct current circuits. Measure $I_{\max}$ and $V_{\max}$ at different frequencies and verify that the impedance is given by

$$Z(\omega) = \sqrt{R^2 + (\omega L - 1/\omega C)^2}. \tag{4.24}$$

For what value of ω is Z a minimum? Note that the relation $V = IZ$ holds only for the maximum values of I and V and not for I and V at any time.

d. Compute the phase difference ϕ_R between the voltage drop across the resistor and the voltage source. Consider $\omega \ll \omega_0$, $\omega = \omega_0$, and $\omega \gg \omega_0$. Does the current lead or lag the voltage in each case, that is, does the current reach a maxima before or after the voltage? Also compute the phase differences ϕ_L and ϕ_C and describe their dependence on ω. Do the relative phase differences between $V_{\rm C}$, $V_{\rm R}$, and $V_{\rm L}$ depend on ω?

4.7 Projects

Project 4.20. Chemical oscillations

The kinetics of chemical reactions can be modeled by a system of coupled first-order differential equations. As an example, consider the following reaction

$$A + 2B \to 3B + C, \tag{4.25}$$

where A, B, and C represent the concentrations of three different types of molecules. The corresponding rate equations for this reaction are

$$\frac{dA}{dt} = -kAB^2 \tag{4.26a}$$
$$\frac{dB}{dt} = kAB^2 \tag{4.26b}$$
$$\frac{dC}{dt} = kAB^2. \tag{4.26c}$$

The rate at which the reaction proceeds is determined by the reaction constant k. The terms on the right-hand side of (4.26) are positive if the concentration of the molecule increases in (4.25) as it does for B and C, and negative if the concentration decreases as it does for A. Note that the term

$2B$ in the reaction (4.25) appears as B^2 in the rate equation (4.26). In (4.26) we have assumed that the reactants are well stirred, so that there are no spatial inhomogeneities. In Section ?? we will discuss the effects of spatial inhomogeneities due to molecular diffusion.

Most chemical reactions proceed to equilibrium, where the mean concentrations of all molecules are constant. However, if the concentrations of some molecules are replenished, it is possible to observe oscillations and chaotic behavior (see Chapter 6). To obtain oscillations, it is essential to have a series of chemical reactions such that the products of some reactions are the reactants of others. In the following, we consider a simple set of reactions that can lead to oscillations under certain conditions (see Lefever and Nicolis):

$$A \to X \tag{4.27a}$$
$$B + X \to Y + D \tag{4.27b}$$
$$2X + Y \to 3X \tag{4.27c}$$
$$X \to C. \tag{4.27d}$$

If we assume that the reverse reactions are negligible and A and B are held constant by an external source, the corresponding rate equations are

$$\frac{dX}{dt} = A - (B+1)X + X^2Y \tag{4.28a}$$
$$\frac{dY}{dt} = BX - X^2Y. \tag{4.28b}$$

For simplicity, we have chosen the rate constants to be unity.

a. The steady state solution of (4.28) can be found by setting dX/dt and dY/dt equal to zero. Show that the steady state values for (X, Y) are $(A, B/A)$.

b. Create a model to solve numerically the rate equations given by (4.28). Your simulation should input the initial values of X and Y and the fixed concentrations A and B, and plot X versus Y as the reactions evolve.

c. Systematically vary the initial values of X and Y for given values of A and B. Are their steady state behaviors independent of the initial conditions?

d. Let the initial value of (X, Y) equal $(A + 0.001, B/A)$ for several different values of A and B, that is, choose initial values close to the steady state values. Classify which initial values result in steady state behavior (stable) and which ones show periodic behavior (unstable). Find the relation between A and B that separates the two types of behavior.

Project 4.21. Nerve impulses

In 1952 Hodgkin and Huxley developed a model of nerve impulses to understand the nerve membrane potential of a giant squid nerve cell. The equations they developed are known as the Hodgkin-Huxley equations. The idea is that a membrane can be treated as a capacitor where $CV = q$ and thus the time rate of change of the membrane potential V is proportional to the current, dq/dt, flowing through the membrane. This current is due to the pumping of sodium and potassium ions through the membrane, a leakage current, and an external current stimulus. The model is capable of producing single nerve impulses, trains of nerve impulses, and other effects. The model is described by the following first-order differential equations:

$$C\frac{dV}{dt} = -g_K n^4(V - V_K) - g_{Na} m^3 h(V - V_{Na}) - g_L(V - V_L) + I_{\text{ext}}(t) \tag{4.29a}$$

$$\frac{dn}{dt} = \alpha_n(1 - n) - \beta_n n \tag{4.29b}$$

$$\frac{dm}{dt} = \alpha_m(1 - m) - \beta_m m \tag{4.29c}$$

$$\frac{dh}{dt} = \alpha_h(1 - h) - \beta_h h, \tag{4.29d}$$

where V is the membrane potential in millivolts (mV), n, m, and h are time dependent functions that describe the gates that pump ions into or out of the cell, C is the membrane capacitance per unit area, the g_i are the conductances per unit area for potassium, sodium, and the leakage current, V_i are the equilibrium potentials for each of the currents, and α_j and β_j are nonlinear functions of V. We use the notation, n, m, and h for the gate functions because the notation is universally used in the literature. These gate functions are empirical attempts to describe how the membrane controls the flow of ions into and out of the nerve cell. Hodgkin and Huxley found the following empirical forms for α_j and β_j:

$$\alpha_n = 0.01(V + 10)/[e^{(1+V/10)} - 1] \tag{4.30a}$$

$$\beta_n = 0.125\, e^{V/80} \tag{4.30b}$$

$$\alpha_m = 0.01(V + 25)/[e^{(2.5+V/10)} - 1] \tag{4.30c}$$

$$\beta_m = 4\, e^{V/18} \tag{4.30d}$$

$$\alpha_h = 0.07\, e^{V/20} \tag{4.30e}$$

$$\beta_m = 1/[e^{(3+V/10)} + 1]. \tag{4.30f}$$

The values of the parameters are $C = 1.0\,\mu\text{F/cm}^2$, $g_K = 36\,\text{mmho/cm}^2$, $g_{Na} = 120\,\text{mmho/cm}^2$, $g_L = 0.3\,\text{mmho/cm}^2$, $V_K = 12\,\text{mV}$, $V_{Na} = -115\,\text{mV}$, and $V_L = 10.6\,\text{mV}$. The unit, mho, represents ohm^{-1}, and the unit of time is milliseconds (ms). These parameters assume that the resting potential of the nerve cell is zero; however, we now know that the resting potential is about $-70\,\text{mV}$.

We can use the ODE solver to solve (4.29) with the state vector $\{V, n, m, h, t\}$; the rates are given by the right-hand side of (4.29). The following questions ask you to explore the properties of the model.

a. Create a model to plot n, m, and h as a function of V in the steady state (for which $\dot{n} = \dot{m} = \dot{h} = 0$). Describe how these gates are operating.

b. Create a model to simulate the nerve cell membrane potential and plot $V(t)$. You can use a simple Euler algorithm with a time step of 0.01 ms. Describe the behavior of the potential when the external current is 0.

c. Consider a current that is zero except for a one millisecond interval. Try a current spike amplitude of $7\,\mu\text{A}$ (that is, the external current equals 7 in our units). Describe the resulting nerve impulse, $V(t)$. Is there a threshold value for the current below which there is no large spike, but only a broad peak?

d. A constant current should produce a train of spikes. Try different amplitudes for the current and determine if there is a threshold current and how the spacing between spikes depends on the amplitude of the external current.

e. Consider a situation where there is a steady external current I_1 for 20 ms and then the current increases to $I_2 = I_1 + \Delta I$. There are three types of behavior depending on I_2 and ΔI. Describe the behavior for the following four situations (1) $I_1 = 2.0\,\mu$A, $\Delta I = 1.5\,\mu$A; (2) $I_1 = 2.0\,\mu$A, $\Delta I = 5.0\,\mu$A; (3) $I_1 = 7.0\,\mu$A, $\Delta I = 1.0\,\mu$A; and (4) $I_1 = 7.0\,\mu$A, $\Delta I = 4.0\,\mu$A. Try other values of I_1 and ΔI as well. In which cases to you obtain a steady spike train? Which cases produce a single spike? What other behavior do you find?

f. Once a spike is triggered, it is frequently difficult to trigger another spike. Consider a current pulse at $t = 20$ ms of $7\,\mu$A that lasts for one millisecond. Then give a second current pulse of the same amplitude and duration at $t = 25$ ms. What happens? What happens if you add a third pules at 30 ms?

4.8 Simulations

The following models are implemented in *EJS* and are downloadable from the OSP Collection in the comPADRE digital library.

SHO Solver Comparison

The SHO Solver Comparison model solves the simple harmonic oscillator (SHO) differential equation using various fixed step size solution algorithms. The simulation allows users to change the step size and uses radio buttons to select the algorithm: Euler, Euler-Richardson, Verlet, and 4th order Runge-Kutta. The output graph in the main window displays the analytic and numerical solution and the Data Tool can be used to compare multiple solutions. See Section 4.1.

Traveling Wave

The Traveling Wave model plots $A\sin(kx + \omega t)$ from $x = x_{\min}$ to $x = x_{\max}$ as a function of t. This *wave function* has both spatial and temporal oscillatory motion and the model shows how how to plot such a time-varying function. See Section 4.1.

Simple Pendulum

The Simple Pendulum model solves the pendulum differential equation using a 4th order Runge-Kutta algorithm. The model is set up and solved using polar coordinates and the view displays the pendulum using polar coordinate axes. This model also demonstrates how to draw the pendulum using as a group of objects and how to rotate the group. See Section 4.2.

Driven SHO Comparison

The Driven SHO Comparison model displays fifty-one oscillators with different natural frequencies driven by an external force. The oscillator mass increases from left to right in the multi-oscillator display and the oscillator in the center has a mass of one. All oscillators are driven with a synchronous external sinusoidal force. The model displays each oscillator as an element of an set and the evolution workpanel for this model shows how *EJS* solves arrays of differential equations. See Section 4.4.

SHO Frequency Response

The SHO Frequency Response model computes a resonance curve by plotting the SHO steady state amplitude as a function of drive frequency. The ODE solver advances the dynamical variables for many drive cycles using the Cash-Karp adaptive step algorithm with a solution tolerance of 10^{-4}. The simulation then computes the amplitude of oscillation and increments the frequency in preparation for the next evolution step. See Section 4.5.

RC Circuit

The RC Circuit model simulates a resistor and a capacitor in series with either a sinusoidal or a square wave source voltage $V_s(t)$ and plots the time dependence of the voltage drops across the source, the resistor, and the capacitor. The resistance, capacitance, and source frequency can be varied by the user. See Section 4.6.

RLC Circuit

The RLC Circuit model simulates a resistor, a capacitor, and an inductor in series with either a sinusoidal or a square wave source voltage $V_s(t$ and plots the time dependence of the voltage drops across the source, the resistor, and the capacitor. The resistance, capacitance, and source frequency can be varied by the user. See Section 4.6.

Truck Drawing

The Truck Drawing model shows how to group multiple elements so that the group can be moved, resized, and rotated as a single object. See Appendix 4A.

Appendix 4A: Compound Elements

Drawings created using the basic graphical elements provided by *EJS* can be moved and resized using the mouse or by linking their position and size properties to model variables. We now describe how to group multiple elements, such as the pendulum group in Section 4.2, so that the group behaves as a single object. We also introduce transformations that can be applied to elements to produce effects such as rotation and shear.

A Group is an empty *EJS* drawing element that can host other elements including other groups. A group has a position and size and accepts a transformation that affects every object within the group. The Truck Drawing model in this chapter's source code directory shows how to create a group. The truck is represented as a rectangular block with two wheels and a windshield. Each of these four parts is displayed using a 2D shape, . In order to move the truck as a whole we could move each of these shapes separately, but this would involve computations for the location of each element. These computations become even more difficult if we were to rotate the truck so as to simulate motion on an inclination plane. The elegant solution is to create a group.

Load the Truck Drawing model in the chapter's directory and inspect the view's tree of elements (left of Figure 4.10), we see that we have created four Shape2D elements inside a Group2D element.

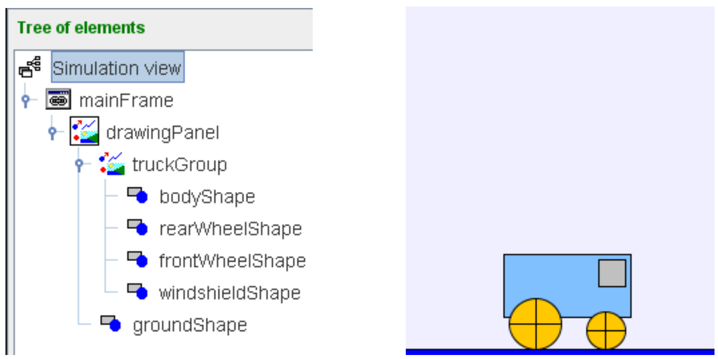

Figure 4.10: A cart is created by grouping other drawables elements.

Elements within a group have a position property relative to the group's position. Inspect the group and note that its position properties are bound to global variables. Inspect the truck body and note that this shape and note that its position properties are constant but that it is draggable and that the Drag Group property is true. Dragging the truck body will affect group properties, not the truck body properties. The truck body Sensitivity property determines the pixel size of the draggable spot within the shape. The special value of zero is used to indicate that the hot spot should fill the entire shape. Run the model and observe that all elements keep their relative positions and sizes as you drag the truck. Note also that the wheels rotate as you drag the truck. This rotation is produced using the wheel's Transform property.

An important property of drawing elements is that of transformations and the most common 2D transformation is a rotation. A Transform property that is set to a double or an integer expression produces a rotation about the element's xy-position. The front and rear wheels in the truck, for example, rotate about the wheel center. This rotation angle is computed using an expression that depends on the x-position and the size of the wheel.

Affine transformations are linear mappings that preserve the *straightness* and *parallelness* of lines. The most common examples of affine transformations are rotations, translations, scales, flips, shears, and their combinations. Use the custom editor icon to the right of the Transform properties field to manipulate elements using these transformations.

The most general 2D transformations are instances of the `java.awt.geom.AffineTransform` class. As specified in the Java documentation, general affine transformations can be represented by a 3 row by 3 column matrix with an implied last row of [0 0 1]. This matrix transforms source coordinates (x,y) into destination coordinates (x',y') by multiplying the coordinate vector by the a matrix:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00}x + m_{01}y + m_{02} \\ m_{10}x + m_{11}y + m_{12} \\ 1 \end{bmatrix}.$$

General transformation objects are created using standard Java syntax and can be used as inspector Transform properties.

```
transformation = new java.awt.geom.AffineTransform(m00, m10, m01, m11, m02, m12);
```

Exercise 4.22. Drawing transformations

Use the transformation properties editor to test the effect of different transformations or combination of transformations on the truck. When you rotate the truck, what is the turning point around which it rotates?

Appendix 4B: ODE Arrays

The Driven SHO Comparison model contains two Evolution workpanels showing how to code arrays of differential equations. The default workpanel expresses a differential equation array using component syntax.

```
d x[i] / dt = v[i]
d v[i] / dt = -k*y[i]/m[i]-b*v[i]/m[i]+externalForce(t)/m[i]
```

Java array components are accessed using integer variables and we often choose the character "i". Note that index variables within an ODE workpanel should not be defined in a global variable table. The index is a local variable that is automatically created by EJS when the ODE workpanel is converted into Java code. You may, of course, use the index within the rate expression to select a dynamical variable array element. Simple derivatives such as

```
dx[i] / dt = -g
```

can be specified without an array index and the expression will be used for all the indexes of the array.

The second Evolution workpanel uses array (vector) syntax. Disable the component notation workpanel and enable the array notation workpanel to test this syntax. Because state values, such as y and v, are stored in arrays we must provide an array or a valid Java expression such as custom method that returns an array. In the array notation workpanel we write

```
d x / dt = v
d v / dt = a
```

The acceleration array is created (instantiated) in the preliminary code pate and the Prelim code button has an exclamation icon to indicate that preparatory code is available and will be executed before the ODE rate is evaluated. Click on this button to see how the acceleration vector a is initialized. Note that the preliminary code loops over all oscillators to compute the acceleration for each element `v[i]`. Although component notation is easier to understand for this simple model, array notation is convenient when the force computation cannot be carried out in a single Java expression. We often use preliminary code and array syntax when many vector forces act on each particle such as in gravitational N-body models.

References and Suggestions for Further Reading

F. S. Acton, *Numerical Methods That Work*, The Mathematical Association of America (1999), Chapter 5.

G. L. Baker and J. P. Gollub, *Chaotic Dynamics: An Introduction*, second edition, Cambridge University Press (1996). A good introduction to the notion of phase space.

Eugene I. Butikov, "Square-wave excitation of a linear oscillator," Am. J. Phys. **72**, 469–476 (2004).

A. Douglas Davis, *Classical Mechanics*, Saunders College Publishing (1986). The author gives simple numerical solutions of Newton's equations of motion. Much emphasis is given to the harmonic oscillator problem.

S. Eubank, W. Miner, T. Tajima, and J. Wiley, "Interactive computer simulation and analysis of Newtonian dynamics," Am. J. Phys. **57**, 457–463 (1989).

Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics*, Vol. 1, Addison-Wesley (1963). Chapters 21 and 23–25 are devoted to various aspects of harmonic motion.

A. P. French, *Newtonian Mechanics*, W. W. Norton & Company (1971). An introductory level text with a good discussion of oscillatory motion.

M. Gitterman, "Classical harmonic oscillator with multiplicative noise," Physica A **352**, 309–334 (2005). The analysis is analytical and at the graduate level. However, it would be straightforward to reproduce most of the results after you learn about random processes in Chapter **??**.

A. L. Hodgkin and A. F. Huxley, "A quantitative description of ion currents and its applications to conduction and excitation in nerve membranes," J. Physiol. (Lond.) **117**, 500–544 (1952).

Charles Kittel, Walter D. Knight, and Malvin A. Ruderman, *Mechanics*, second edition, revised by A. Carl Helmholz and Burton J. Moyer, McGraw-Hill (1973).

R. Lefever and G. Nicolis, "Chemical instabilities and sustained oscillations," J. Theor. Biol. **30**, 267 (1971).

Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics*, fifth edition, Harcourt (2004). Excellent discussion of linear and nonlinear oscillators.

M. F. McInerney, "Computer-aided experiments with the damped harmonic oscillator," Am. J. Phys. **53**, 991–996 (1985).

William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical Recipes*, second edition, Cambridge University Press (1992). Chapter 16 discusses the integration of ordinary differential equations.

Scott Hamilton, *An Analog Electronics Companion*, Cambridge University Press (2003). A good discussion of the physics and mathematics of basic circuit design including an extensive introduction to circuit simulation using the PSpice simulation program.

S. C. Zilio, "Measurement and analysis of large-angle pendulum motion," Am. J. Phys. **50**, 450–452 (1982).